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ON GENERATION OF THE GROUPS $GL_n(q)$ and $PGL_n(q)$ BY THREE INVOLUTIONS, TWO OF WHICH COMMUTE

I. A. Markovskaya, Ya. N. Nuzhin

A group generated by three involutions, two of which commute, will be called $(2 \times 2, 2)$-generated. The class of such groups is closed with respect to homomorphic images, if, by definition, we consider the identity group as such and do not exclude the coincidence of two or all three involutions. For finite simple groups, the question of generation by three involutions, two of which commute, was formulated by V. D. Mazurov in the Kourovka notebook in 1980. The answer to this question is known, and it is positive, excluding three alternating groups, some groups of Lie type of rank no more than three, and four sporadic groups. This article considers the $(2 \times 2, 2)$-generation of the general linear group over a finite field and its projective image $PGL_n(q)$.
It is proven that $GL_n(q)$ (respectively $PGL_n(q)$) is $(2 \times 2, 2)$-generated if and only if a) $q = 2$ and $n = 2$ or $n \geq 5$, or b) $q = 3$ and $n \geq 5$ (respectively, when either a) $n = 2$ and any q, or b) $n \geq 4$ and $(n, q-1) = 2$, or c) $n \geq 5$ and $(n, q-1) = 1$).

Keywords: general and projective linear groups, finite field, generating triples of involutions.

MSC: 20G40

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Irina Aleksandrovna Markovskaya, doctoral student, Institute of Mathematics and Computer Science of the Siberian Federal University, Krasnoyarsk, 660041 Russia, e-mail: mark.i.a@mail.ru .

Yakov Nifantievich Nuzhin, Dr. Phys.-Math. Sci., Prof. Institute of Mathematics and Computer Science of the Siberian Federal University, Krasnoyarsk, 660041 Russia,
e-mail: nuzhin2008@rambler.ru .

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