

**NONLINEAR ELLIPTIC VARIATIONAL INEQUALITIES  
WITH UNILATERAL POINTWISE FUNCTIONAL CONSTRAINTS  
IN VARIABLE DOMAINS**

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We consider variational inequalities with operators  $\mathcal{A}_s: W^{1,p}(\Omega_s) \rightarrow (W^{1,p}(\Omega_s))^*$  in divergence form and constraint sets  $V_s = \{v \in W^{1,p}(\Omega_s) : h(v) + \Phi_s(v) \leq \varphi_s \text{ a.e. in } \Omega_s\}$ , where  $\Omega_s$  with  $s \in \mathbb{N}$  is a domain in  $\mathbb{R}^n$  contained in a bounded domain  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ),  $p > 1$ ,  $h$  is a convex function on  $\mathbb{R}$ ,  $\varphi_s$  is a function on  $\Omega_s$ , and  $\Phi_s$  is a continuous convex functional on  $W^{1,p}(\Omega_s)$ . We describe conditions for a weak convergence of solutions of the considered variational inequalities to the solution of a variational inequality with an operator from  $W^{1,p}(\Omega)$  to  $(W^{1,p}(\Omega))^*$  and constraint set defined by the equality  $V = \{v \in W^{1,p}(\Omega) : h(v) + \Phi(v) \leq \varphi \text{ a.e. in } \Omega\}$ , where  $\varphi$  is a limit function for  $\varphi_s$  and  $\Phi$  is a limit functional for  $\Phi_s$ . These conditions include some requirements on the involved domains, operators, and the mappings defining the constraint sets. In so doing, one of the main conditions is the  $G$ -convergence of the sequence  $\{\mathcal{A}_s\}$  to an operator  $\mathcal{A}: W^{1,p}(\Omega) \rightarrow (W^{1,p}(\Omega))^*$ .

Keywords: nonlinear elliptic variational inequality, pointwise functional constraint, variable domains,  $G$ -convergence of operators, convergence of solutions.

**MSC:** 47J20, 49J40, 49J45

**DOI:** 10.21538/0134-4889-2025-31-4-132-148

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Received October 15, 2025

Revised October 28, 2025

Accepted November 3, 2025

**Funding Agency:** This work was partially supported by the Ministry of Science and Higher Education of the Russian Federation within the framework of the Program of development of Ural Federal University under the “Priority-2030” academic leadership program.

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Cite this article as: A. A. Kovalevsky. Nonlinear elliptic variational inequalities with unilateral pointwise functional constraints in variable domains. *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2025, vol. 31, no. 4, pp. 132–148.