

## BERNSTEIN INEQUALITY FOR FRACTIONAL POWERS OF UNIVARIATE DUNKL LAPLACIAN AND MULTIVARIATE LAPLACE OPERATOR OF ENTIRE FUNCTIONS

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We consider fractional powers of order  $\alpha > 0$  of the Dunkl Laplacian and the Laplace operator (Riesz derivatives) on classes of entire functions of exponential spherical type of one and many variables. The present study investigates various definitions of these fractional powers, including those involving Fourier (Fourier–Dunkl) multipliers, hypersingular integrals, interpolation formulas containing usual and generalized translations with equidistant and non-equidistant steps. We examine the interrelations between the fractional powers of the Dunkl Laplacian and Riesz derivatives for functions of one and many variables. Sharp Bernstein inequalities for these operators are explored. The study delves into the exploration of sharp Bernstein inequalities for these operators. Earlier, S.S. Platonov (2007) obtained a sharp inequality for the Dunkl Laplacian ( $\alpha = 2$ ) on the set of even functions. O.L. Vinogradov (2023) proved sharp inequalities for the fractional powers of the Dunkl Laplacian and Riesz derivatives of functions of several variables for  $\alpha \geq 1$ . In this paper, we obtain sharp Bernstein inequalities for the fractional powers of the Dunkl Laplacian and Riesz derivatives of functions of several variables for  $0 < \alpha < 1$ . The tools for the proof are interpolation formulas containing generalized Dunkl translations with non-equidistant steps, which are zeros of Bessel functions.

Keywords: Riesz derivative, Dunkl Laplacian, entire functions of exponential spherical type, Bernstein inequality, uniform norm.

MSC: 26A33, 32A15, 41A17

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