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ON EXTREMAL TRIGONOMETRIC POLYNOMIALS

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Let $\mathscr{F}_n$ be the set of all trigonometric polynomials of order $\le n$, $n \in \mathbb{N}$. For multipliers $H : \mathscr{F}_n \to \mathscr{F}_n$, we prove an interpolation formula $H(f)(t) = \sum_{k=0}^{2n-1} \Lambda_k f\left(t - \tau + k\pi/n\right)$, which is used to obtain the following inequalities and criteria for an extremal polynomial in them (Theorem 4):

$$\int_{\mathbb{T}} J\left(|H(f)(t)|\right)\, dt \le \int_{\mathbb{T}} J\left(\varkappa|f(t)|\right)\, dt\,;\;\; \|H(f)\|_p \leqslant \varkappa\|f\|_p,\, 1 \le p \le \infty,\, \varkappa = |\Lambda_0| + \ldots + |\Lambda_{2n-1}| > 0.$$

Here the function J is convex and nondecreasing on $[0, +\infty)$. The main goal of this work is to describe all extremal polynomials in the above inequalities. Theorem 5 proves that if the function J is convex and strictly increasing on $[0, +\infty)$ and two conditions are satisfied: (1) $\exists s \in \mathbb{Z} : \overline{\Lambda_s}\Lambda_{s+1} < 0$ and (2) $\exists \varepsilon \in \mathbb{C}$, $|\varepsilon| = 1 :$ $\varepsilon\Lambda_k(-1)^k \ge 0$, $k \in \mathbb{Z}$, then only polynomials of the form $f(t) = \mu e^{int} + \nu e^{-int}$, $\mu, \nu \in \mathbb{C}$ are extremal in these inequalities. The main cases in this theorem are the cases $p = \infty$ and $p = 1$. Theorem 6 proves that if the function J is convex and strictly increasing on $[0, +\infty)$ and the operator H satisfies the Szegö condition (the nonnegativity of a special trigonometric polynomial), then, in all cases different from one exceptional case, only polynomials of the form $f(t) = \mu e^{int} + \nu e^{-int}$, $\mu, \nu \in \mathbb{C}$, are extremal in these inequalities. In the exceptional case, there are other extremal polynomials. In this paper we give general examples of operators H that satisfy the conditions of Theorem 6 (Example 1, Theorems 7 and 8). In particular, S. T. Zavalishchin's operator (Example 2) and the fractional derivative operator $H(f)(t) = f^{(r,\beta)}(t)$, $\beta \in \mathbb{R}$, $r \ge 1$, $\varkappa = n^r$ (Corollary 3), satisfy these conditions. In this paper we also describe extremal polynomials in the Trigub and Boas inequalities (for some values of the parameters, not only polynomials of the form $\mu e^{int} + \nu e^{-int}$ are extremal).

Keywords: extremal trigonometric polynomial, Bernstein condition, Szegö condition, Weil–Nagy derivative, Bernstein-Szegö inequality, positive definite function, Boas–Civin method.

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