

**BEST APPROXIMATION OF A FRACTIONAL-ORDER DIFFERENTIATION
OPERATOR IN THE UNIFORM NORM ON THE AXIS ON THE CLASS
OF FUNCTIONS WITH INTEGRABLE FOURIER TRANSFORM
OF THE HIGHEST DERIVATIVE**

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A solution is given to Stechkin's problem on the best approximation in the uniform norm on the real axis of differentiation operators of fractional (more precisely, real) order k by bounded linear operators from the space L^2 to the space C on the class of functions \mathcal{Q}^n whose Fourier transform of the n th-order fractional derivative, $0 \leq k < n$, is integrable. The corresponding exact Kolmogorov inequality is given. A solution is obtained to the problem of optimal recovery of the differentiation operator of fractional order k on functions of the class \mathcal{Q}^n defined with a known error in the space L^2 .

Keywords: fractional-order differentiation operator, Stechkin's problem, Kolmogorov inequality, optimal differentiation.

MSC: 47B38, 54C35, 47A58, 26D10

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