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FINDING THE VALUE OF THE CHEBYSHEV LAYER OF A FLAT SET USING CONSTRUCTIONS OF THE THEORY OF ALPHA SETS AND EFIMOV–STECHKIN SUPPORT BALLS

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For a class of closed nonconvex sets in two-dimensional Euclidean space, an approach to finding the value of the Chebyshev layer is proposed. It is based on two well-known concepts that generalize the definition of a convex set. A family of planar sets with a finite number of pseudo-vertices is considered. Three sets of pseudo-vertices are selected for analysis. The sets differ from each other in the order of smoothness of the pseudo-vertices included in them. Within the framework of each of the three cases considered (the case of a piecewise smooth boundary of a set, the case of a discontinuity in the curvature of the boundary of a set, and the classical case when the curvature of the boundary is continuous), a formula for the limit value of the radii of the support balls (by Efimov and Stechkin) is found. We consider balls with centers lying on a branch of the bisector (on a one-dimensional manifold of the set of non-uniqueness) corresponding to the associated pseudo-vertex. The obtained formulas allow one to analytically calculate the value of the Chebyshev layer for nonconvex sets, including sets with a boundary of variable smoothness. An illustrative example and its interpretation from the point of view of optimal control theory are given.

Keywords: alpha set, set hull, metric projection, nonconvexity measure, set bisector, support ball, Chebyshev layer, control.

MSC: 52A10, 11H16, 41A50

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