

PROJECTION CLOSED SETS AND EFIMOV–STECHKIN SPACES

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The paper is concerned with problems of existence, uniqueness, and stability of best approximants and related problems of solarity for not necessarily closed sets (in particular, such sets are not necessarily proximal). We give definitions of the projection boundary and of a projection closed set. We show that if X is a symmetrizable asymmetric Efimov–Stechkin space, then the set of points of approximative compactness of any nonempty projection closed set $M \subset X$ is of second category in the metric exterior $\text{out}(M)$ of the set M . We also study relations between classes of sets in asymmetric Efimov–Stechkin spaces. We show that if X is a symmetrizable Efimov–Stechkin space, $M \subset X$ is projection closed, and $\ell_0 P_0$ -connected, then M is $\ell_0 \tilde{B}$ -connected. Solarity problem for sets of uniqueness is studied. In particular, the well-known theorem dating back to V. I. Berdyshev and V. L. Klee on solarity of boundedly compact Chebyshev sets is extended to the case of sets of uniqueness. Results on solarity of boundedly precompact projection closed sets of uniqueness are obtained. We also obtain results on preservation of solarity and other approximative properties of sets when changing to closed neighborhoods $M + B(x, r)$ of sets. Special attention is given to the case of Chebyshev suns, for which a characterization theorem is obtained.

Keywords: projection boundary, projection closed set, connected set, best approximation, sun, Chebyshev sun, Efimov–Stechkin space, metrical exterior of a set, approximative compactness, approximative precompactness.

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REFERENCES

1. Braess D. *Nonlinear approximation theory*. Springer Ser. Comput. Math, vol. 7. Berlin, Heidelberg, Springer, 1986, 290 p. <https://doi.org/10.1007/978-3-642-61609-9>
2. Alimov A.R., Tsar'kov I.G. *Geometric approximation theory*. Springer Monogr. Math. Ser. Cham, Springer, 2021, 508 p. <https://doi.org/10.1007/978-3-030-90951-2>
3. Efimov N.V., Stechkin S.B. Some properties of Chebyshev sets. *Dokl. Akad. Nauk SSSR*, 1958, vol. 118, no. 1, pp. 17–19 (in Russian).
4. Efimov N.V., Stechkin S.B. Chebyshev sets in Banach spaces. *Dokl. Akad. Nauk SSSR*, 1958, vol. 121, no. 4, pp. 582–585 (in Russian).
5. Efimov N.V., Stechkin S.B. Support properties of sets in Banach spaces and Chebyshev sets. *Dokl. Akad. Nauk SSSR*, 1959, vol. 127, no. 2, pp. 254–257 (in Russian).
6. Efimov N.V., Stechkin S.B. Approximate compactness and Chebyshev sets. *Soviet Math. Dokl.*, 1961, vol. 2, pp. 1226–1228 (in Russian).
7. Alimov A.R., Tsar'kov I.G. Connectedness and approximative properties of sets in asymmetric spaces. *Filomat*, 2024, vol. 38, no. 9, pp. 3243–3259. <https://doi.org/10.2298/FIL2409243A>
8. García-Raffi L.M., Romaguera S., Sánchez Pérez E.A. On Hausdorff asymmetric normed linear spaces. *Houston J. Math.*, 2003, vol. 2, no. 3, pp. 717–728.
9. Cobzaş S. Compact bilinear operators on asymmetric normed spaces. *Topology Appl.*, 2022, vol. 306, art. no. 107922, 23 p. <https://doi.org/10.1016/j.topol.2021.107922>
10. Alimov A.R., Tsar'kov I.G. Stability of best approximation in classical problems of geometric approximation theory. *Izv. RAN. Ser. Mat.*, 2025 (in Russian). <https://doi.org/10.4213/im9660>
11. Borwein J.M., Fitzpatrick S. Existence of nearest points in Banach spaces. *Canad. J. Math.*, 1989, vol. 41, no. 4, pp. 702–720. <https://doi.org/10.4153/CJM-1989-032-7>
12. Tsar'kov I.G. Properties of at most countable unions of pairwise disjoint sets in asymmetric spaces. *Sb. Math.*, 2025, vol. 216, no. 2, pp. 257–269. <https://doi.org/10.4213/sm10104e>

13. Singer I. Some remarks on approximative compactness. *Rev. Roum. Math. Pures Appl.*, 1964, vol. 9, no. 2, pp. 167–177.
14. Alimov A.R., Tsar'kov I.G. Approximatively compact sets in asymmetric Efimov–Stechkin spaces and convexity of almost suns. *Math. Notes*, 2021, vol. 110, no. 6, pp. 947–951.
<https://doi.org/10.1134/S0001434621110316>
15. Tsar'kov I.G. Relations between certain classes of sets in Banach spaces. *Math. Notes*, 1986, vol. 40, no. 2, pp. 597–610. <https://doi.org/10.1007/BF01159114>
16. Alimov A.R., Tsar'kov I.G. Connectedness and solarity in problems of best and near-best approximation. *Russian Math. Surv.*, 2016, vol. 71, no. 1, pp. 1–77. <https://doi.org/10.1070/RM9698>
17. Stechkin S.B. Approximative properties of sets in normed linear spaces. *Rev. Math. Pures Appl.*, 1963, vol. 8, no. 1, pp. 5–18 (in Russian).
18. Konyagin S.V. On approximation properties of closed sets in Banach spaces and the characterization of strongly convex spaces. *Dokl. Akad. Nauk SSSR*, 1980, vol. 251, no. 2, pp. 276–280 (in Russian).
19. Konyagin S.V. *Approximative properties of sets in normed linear spaces*. Dno. Cand. Phys.-Math. Sci., Moscow, MGU, 1982, 100 p. (in Russian).
20. Karlov M.I. Approximative properties of compact C^2 -manifolds in Hilbert space. *East J. Approx.*, 1996, vol. 2, pp. 197–204.
21. Alimov A.R., Tsar'kov I.G. \tilde{B} -complete sets: approximative and structural properties. *Sib. Math. J.*, 2022, vol. 63, pp. 412–420. <https://doi.org/10.1134/S0037446622030028>
22. Berdyshev V.I. On Chebyshev sets. *Dokl. Akad. Nauk AzSSR*, 1966, vol. 22, no. 9, pp. 3–5 (in Russian).
23. Klee V.L. Convex bodies and periodic homeomorphism in Hilbert space. *Trans. Amer. Math. Soc.*, 1953, vol. 4, pp. 10–43.
24. Koshcheev V.A. The connectivity and approximative properties of sets in linear normed spaces. *Math. Notes*, 1975, vol. 17, no. 2, pp. 114–119. <https://doi.org/10.1007/BF01161866>
25. Koshcheev V.A. Some properties of the δ -projection in normed linear spaces. *Sov. Math. (Iz. VUZ)*, 1976, vol. 20, no. 5, pp. 26–30.
26. Vlasov L.P. Approximative properties of sets in normed linear spaces. *Rus. Math. Surv.*, 1973, vol. 28, no. 6, pp. 1–66. <https://doi.org/10.1070/RM1973v028n06ABEH001624>
27. Pyatyshev I.A. Operations on approximatively compact sets. *Math. Notes*, 2007, vol. 82, no. 5, pp. 653–659. <https://doi.org/10.1134/S0001434607110089>
28. Alimov A.R., Karlov M.I. Sets with external Chebyshev layer. *Math. Notes*, 2001, vol. 69, no. 2, pp. 269–273. <https://doi.org/10.1023/A:1002836705675>
29. Balaganskii V.S. Approximative properties of sets in Hilbert space. *Math. Notes*, 1982, vol. 31, no. 5, pp. 397–404. <https://doi.org/10.1007/BF01145720>

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