

**PARAMETRIC FAMILIES OF REGULARIZERS FOR PRODUCTS OF
ELEMENTARY GENERALIZED FUNCTIONS****A. V. Makarov**

V.K. Ivanov, in a series of works, constructed a real, associative, commutative, differential algebra generated by elementary distributions (generalized functions) with singularities at the origin. The values of products for $x \neq 0$ remain unchanged. Following ideas of S.L. Sobolev, M. Sato, and G. Bremermann, each distribution is associated with its Poisson representation, which is a harmonic function in the upper half-plane. The product of harmonic functions is harmonic only in rare cases. In the algebra of products of harmonic functions corresponding to elementary distributions, a multiplicative homomorphism (a regularizer) is constructed that assigns to the product of harmonic functions a harmonic function which is the Poisson representation of some elementary generalized function. Thus a product of distributions is defined. Moreover, it is proved that such a homomorphism is unique. In the present work, a parametric family of regularizers is constructed that generates a real, commutative, differential algebra of elementary distributions with singularities at the origin. Associativity of products and preservation of values for $x \neq 0$ are not assumed. Relations are obtained between the products of elementary generalized functions and distributions.

Keywords: products of elementary generalized functions, Poisson transform, regularizer.

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