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INTERNAL STRUCTURE OF CONVEX SETS AND THEIR FACES

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Most often, the geometric structure of convex sets is associated with their facial structure. In the first section of this paper, we present a somewhat different approach to characterizing the geometric structure of convex sets based on the concept of an open component of a convex set. In this paper, we consider convex sets in infinite-dimensional real vector spaces endowed with no topology. To define the notion of an open component of a convex set Q, the preorder relation $\trianglelefteq_Q$ is introduced on Q (its own for each set Q) called a dominance relation. Open components of a convex set Q are defined as equivalence classes of the quotient set $Q/{<\!\!>_Q}$ of the set Q by the equivalence relation $<\!\!>_Q$, which is the symmetric part of the dominance relation $\trianglelefteq_Q$. Each open component of a convex set Q is a relatively algebraic open subset of the set Q under consideration, and the set Q is a disjoint union of all open components belonging to Q. The dominance relation $\trianglelefteq_Q$ induces a partial order relation $\trianglelefteq^*_Q$ on the family $\mathcal{O}(Q) := Q/{<\!\!>_Q}$ of all open components of the set Q with respect to which the partially ordered family $(\mathcal{O}(Q), \trianglelefteq^*_Q)$ is an upper semilattice. For halfspaces (convex sets H whose complements are also convex), the corresponding upper semilattice $(\mathcal{O}(H), \trianglelefteq^*_H)$ is a linearly ordered set. The internal structure of a convex set Q is identified in the paper with the structure of the upper semilattice $(\mathcal{O}(Q), \trianglelefteq^*_Q)$. In the second section of the paper, the connection between the internal structure of a convex set and that of its faces is investigated. It is established that each open component of a convex set Q is a relative algebraic interior of the minimal (with respect to inclusion) face of Q containing the given open component. Conversely, if a face F of a convex set Q has a nonempty relative algebraic interior, then it (the relative algebraic interior of the face) coincides with some open component of the set Q, and the face F itself is a minimal face containing this open component (such faces are called minimal in the paper). In finite-dimensional vector spaces, any face F of a convex set Q is minimal, whereas in any infinite-dimensional vector space, there exist convex sets whose faces are not all minimal. Concurrently, each open component of any face F of a convex set Q is an open component of Q itself; i.e., $\mathcal{O}(F) \subset \mathcal{O}(Q)$. Moreover, the partial order relation $\trianglelefteq^*_F$ defined on $\mathcal{O}(F)$ coincides with the restriction to $\mathcal{O}(F)$ of the partial order relation $\trianglelefteq^*_Q$ defined on $\mathcal{O}(Q)$. Thus, the internal structure $(\mathcal{O}(F), \trianglelefteq^*_F)$ of any face F of a convex set Q is a substructure of the internal structure $(\mathcal{O}(Q), \trianglelefteq^*_Q)$ of Q itself.

Keywords: convex sets, halfspaces, faces, open component, semilattice, preorder, linear order.

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