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ON COMPLEMENTARITY AND LINEAR HOMEOMORPHISM OF $C_P(X)$ SPACES FOR COUNTABLE METRIC SPACES X

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In this paper, we consider the complementarity of the space $C_p(X)$ in the space $C_p(Y)$ for countable sparse metrizable spaces X and Y. It is said that the space $C_p(X)$ is complementably embedded in the space $C_p(Y)$ if there exists a linear homeomorphism $C_p(X)$ to the complemented subspace $L \subset C_p(Y)$. We prove that if for some ordinal α the derivative $X^{(\alpha\cdot\omega)} \neq \varnothing$, and $Y^{(\omega)} = \varnothing$, then the space $C_p(X)$ is complementably not embedded in the space $C_p(Y)$. We also consider the derivatives $X^{(\alpha)}$, which are defined similarly to $X^{(\alpha)}$ by removing all points having a compact neighborhood. It is proved that if $X^{\{\alpha\}} \neq \varnothing$, and $Y^{\{\alpha\}} = \varnothing$, then the space $C_p(X)$ is not complementably embedded in the space $C_p(Y)$. Futhermore, if $X^{\{\alpha\}} = Y^{\{\alpha\}} = \varnothing$, $X^{\{\alpha-1\}}$ is a locally compact non-compact space, and $Y^{\{\alpha-1\}}$ is compact, then the space $C_p(X)$ is complementably not embedded in the space $C_p(Y)$. For the proof, the method of decomposition of the space $C_p(X)$ into a countable product of the spaces $C_p(X_n)$ and the existence of a continuous linear extension operator $T : C_p(L) \longrightarrow C_p(X)$ for a closed subset of $L \subset X$.

Keywords: homeomorphism, linear homeomorphism, topology of pointwise convergence, retract, projector, complemented subspaces, ordinal, closed graph theorem.

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