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ON ESTIMATES OF THE APPROXIMATION OF FUNCTIONS FROM A SYMMETRIC SPACE BY FOURIER SUMS IN THE UNIFORM METRIC

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The article discusses the symmetric space of periodic functions of several variables, specifically, the generalized Lorentz–Zygmund space and the Nikol'skii–Besov class within this space. Estimates for the approximation of functions from the Nikol'skii–Besov class by partial sums over step hyperbolic crosses of Fourier series are established in the uniform metric. An analog of the Jackson–Nikol'skii inequality for multiple trigonometric polynomials in the norms of the generalized Lorentz–Zygmund space and the space of continuous functions is proved.

Keywords: symmetric space, Fourier sum, Nikol'skii–Besov class, Lorentz–Zygmund space.

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