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4-GRACEFUL TREES

V. A. Baransky, I. A. Nasyrov, T. A. Sen'chonok

Let f be a coloring of the vertices of a connected graph G into colors from a set $\{1, 2, \ldots, k\}$. The coloring f induces on the set of edges of G a function $f'(e) = |f(u) - f(v)|$, where $e = uv$ is an arbitrary edge of G. The integer $f'(e)$ will be called the *color* of the edge e *induced by the coloring* f. The coloring f is called a *graceful k-coloring* of G if f' is an edge coloring of this graph. We will call a graph *k-graceful* or, simply, *graceful* if it has a graceful k-coloring. The main goal of our work is to give two sufficient conditions, each of which guarantees the 4-gracefulness of trees (see Theorems **??** and **??**). In addition, we study the properties of 4-graceful graphs to prove these theorems. We also plan to use the found properties in the subsequent works on 4-graceful graphs and 4-graceful trees in the general case. Let us note that the 4-graceful trees are quite complex, while the 3-graceful connected graphs have a very simple structure. The latter are limited to simple chains of length at most 3. It is easy to establish that any 4-graceful graph does not contain vertices of degree greater than 3. Therefore, we consider trees T whose vertex degrees do not exceed 3. We will call vertices of degree 3 in a tree T *nodes*. For two nodes u and v of a tree T, we will say that the *nodal distance* between them is t if there is a simple chain from u to v such that the number of internal vertices of this chain is $t \geq 1$ and all these internal vertices have degree 2 in the tree T. Of course, the nodal distance is not always defined for a pair of nodes. Among other properties of 4-graceful graphs, it established that such graphs do not contain simple chains consisting of three nodes. The sufficient conditions for 4-gracefulness formulated in the paper consist in prohibiting the appearance in trees of pairs of nodes whose nodal distances are equal to 1 or 2.

Keywords: graph, tree, graceful coloring of a graph, graceful trees.

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Vitaly Anatol'evich Baransky, Dr. Phys.-Math. Sci., Prof., Ural Federal University, Yekaterinburg, 620000 Russia, e-mail: vitaly.baransky@urfu.ru .

Ilia Aleksandrovich Nasyrov, PhD student, Ural Federal University, Yekaterinburg, 620000 Russia, e-mail: ilia.nasyrov@urfu.ru .

Tatiana Aleksandrovna Sen'chonok, Cand. Sci. (Phys.-Math.), Ural Federal University, Yekaterinburg, 620000 Russia, e-mail: tatiana.senchonok@urfu.ru .

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