

MSC: 47B38, 47A58, 26D10

DOI: 10.21538/0134-4889-2024-30-4-37-54

## A VARIANT OF STECHKIN'S PROBLEM ON THE BEST APPROXIMATION OF A FRACTIONAL ORDER DIFFERENTIATION OPERATOR ON THE AXIS

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A solution is given to Stechkin's problem on the best approximation on the real axis of differentiation operators of fractional (more precisely, real) order  $k$  in the space  $L_2$  by bounded linear operators from the space  $L$  to the space  $L_2$  on the class of functions whose fractional derivative of order  $n$ ,  $0 \leq k < n$ , is bounded in the space  $L_2$ . An upper estimate is obtained for the best constant in the corresponding Kolmogorov inequality. It is shown that the well-known Stechkin lower estimate for the value of the problem of approximating the differentiation operator via the best constant in the Kolmogorov inequality is strict in this case; in other words, Stechkin's problem and the Kolmogorov inequality are not consistent.

Keywords: fractional order differentiation operator, Stechkin's problem, Kolmogorov inequality, Carlson inequality.

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Received June 19, 2024

Revised September 17, 2024

Accepted September 23, 2024

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Cite this article as: V.V. Arestov. A variant of Stechkin’s problem on the best approximation of a fractional order differentiation operator on the axis. *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2024, vol. 30, no. 4, pp. 37–54.