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STRICT SUNS COMPOSED OF PLANES

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A set M is a strict sun if, for each $x \notin M$, the set $P_M x$ of best approximants from M for x is nonempty and each point $y \in P_M x$ is a nearest point from M for each point z from the ray emanating from y and passing through x. Strict suns are sometimes called Kolmogorov sets, because they satisfy the Kolmogorov criterion for best approximation. We study structural properties of strict suns composed of a finite number of planes (affine spaces, which may possibly degenerate to points). We always assume that the union of planes $M := \bigcup L_i$ is irreducible, i.e., no plane in this union contains another plane from the union. We show that if an irreducible finite union of planes $M := \bigcup_{i=1}^N L_i$ is a strict sun in a normed space, then M consists of a single plane. In this result, the strict sun cannot be replaced by a sun. A stronger local analog of this result is proved in the space ℓ^∞_n. Namely, we show that if $M := \bigcup_{i=1}^N L_i$ is an irreducible union of planes in ℓ^∞_n, Π is a bar (intersection of extreme hyperplanes), and $M \cap \Pi \neq \varnothing$, then $M' := M \cap \Pi$ is a strict sun in ℓ^∞_n if and only if M' is convex, i.e., M' is the intersection of some plane L_i with the bar Π. As a corollary, if $M := \bigcup_{i=1}^N L_i$ is a local strict sun in ℓ^∞_n, then M consists of a single plane. Similar results are also established for sets $M := \bigcup_{i=1}^N L_i$ with continuous metric projection in ℓ^∞_n. The present paper continues and develops the previous studies on approximation by Chebyshev sets composed of planes began by the author of the article and I.G. Tsar'kov in linear normed and asymmetrically normed spaces and the results of I.G. Tsar'kov on sets with piecewise continuous metric projection.

Keywords: best approximation, union of planes, sun, strict sun, discretization.

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