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SOLUTIONS WITH A ZERO FRONT TO THE QUASILINEAR PARABOLIC HEAT EQUATION

A. L. Kazakov

The paper considers a nonlinear second-order evolution equation known as the nonlinear (quasilinear) heat equation with a source (sink) and also as the generalized porous medium equation. We deal with the case of an arbitrary dimension, and there is central (axial) symmetry. In other words, the unknown function depends on time t and the distance ρ to some point (straight line). The study concerns nontrivial solutions with a zero front that describe disturbances propagating over a stationary (cold) background with a finite velocity. A new theorem for the existence and uniqueness of a solution with the desired properties is proved. It allows one to construct the solution as a special series with recursively calculated coefficients and to cancel the singularity at the point $\rho = 0$ by a degenerate change of independent variables. For the considered problem, an analog of S.V. Kovalevskaya's example is presented. We obtain conditions which ensure that the coefficients of the constructed series are constants; i.e., the original problem is reduced to the integration of an ordinary differential equation with a singularity in the factor at the highest derivative. The properties of the ordinary differential equation are studied using majorant methods and qualitative analysis. The results obtained are interpreted from the point of view of the original problem.

Keywords: nonlinear partial differential equations, generalized porous medium equation, degeneration, initial–boundary value problem, existence and uniqueness theorem, series, convergence, majorant method, exact solution, qualitative analysis of ordinary differential equations.

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Alexander Leonidovich Kazakov, Dr. Phys.-Math. Sci., Prof., Matrosov Institute for System Dynamics and Control Theory of the Siberian Branch of the Russian Academy of Sciences, Irkutsk, 664033 Russia, e-mail: kazakov@icc.ru .

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