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ASYMPTOTIC EXPANSION OF THE ERROR OF A NUMERICAL METHOD FOR SOLVING A SUPERDIFFUSION EQUATION WITH FUNCTIONAL DELAY**V. G. Pimenov, A. B. Lozhnikov**

An equation with Riesz fractional space derivatives and a functional delay effect is considered. The problem is discretized. Constructions of an analog of the Crank–Nicolson difference method with piecewise linear interpolation and extrapolation by continuation are presented. The method has the second order of smallness with respect to the time and space sampling steps Δ and h . The basic Crank–Nicolson method with piecewise parabolic interpolation and extrapolation by continuation is constructed. The order of the residual without interpolation of the basic method is studied. The expansion coefficients of the residual with respect to Δ and h are written. An equation is derived for the main term of the asymptotic expansion of the global error. Under certain assumptions, the legality of using the Richardson extrapolation procedure is substantiated and the corresponding method is constructed. The main of these assumptions is the consistency of the orders of smallness Δ and h . It is proved that the method has order $O(\Delta^3 + h^3)$.

Keywords: Riesz fractional derivatives, superdiffusion equation, functional delay, Crank–Nicolson method, piecewise parabolic interpolation, extrapolation by continuation, Richardson method.

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