NONPRONORMAL SUBGROUPS OF ODD INDEX IN FINITE SIMPLE LINEAR AND UNITARY GROUPS

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A subgroup H of a group G is *pronormal* if, for each $g \in G$, the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$. Most of finite simple groups possess the following property $(*)$: each subgroup of odd index is pronormal in the group. The conjecture that all finite simple groups possess the property $(*)$ was established in 2012 in a paper by E. P. Vdovin and the third author based on the analysis of the proof that Hall subgroups are pronormal in finite simple groups. However, the conjecture was disproved in 2016 by A. S. Kondrat'ev together with the second and third authors. In a series of papers by Kondrat'ev and the authors published from 2015 to 2020, the finite simple groups with the property $(*)$ except finite simple linear and unitary groups with some constraints on natural arithmetic parameters were classified. In this paper we construct series of examples of nonpronormal subgroups of odd indices in finite simple linear and unitary groups over a field of odd characteristic, thereby making a step towards completing the classification of finite simple groups with the property $(*)$.

Keywords: finite group, simple group, linear simple group, unitary simple group, pronormal subgroup, odd index.

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