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QUESTIONS OF EXISTENCE, ABSENCE, AND UNIQUENESS OF A SOLUTION TO ONE CLASS OF NONLINEAR INTEGRAL EQUATIONS ON THE WHOLE LINE WITH AN OPERATOR OF HAMMERSTEIN–STIELTJES TYPE

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The work is devoted to the study of questions of the existence, nonexistence, and uniqueness of a solution to one class of integral equations of the Hammerstein–Stieltjes type on the whole line with a concave and monotone nonlinearity. This class of equations has direct applications in various areas of modern natural science. In particular, depending on the representation of the corresponding kernel (or subkernel) and nonlinearity, equations of this kind are found in probability theory (Markov processes), p-adic string theory, the theory of radiative transfer in spectral lines, epidemiology, and the kinetic theory of gases and plasma. Under certain constraints on the kernel and on the nonlinearity of the equation, a constructive theorem for the existence of a continuous positive bounded solution is proved. A method for constructing an approximate solution is also outlined, the essence of which is to obtain a uniform estimate of the difference between the constructed solution and the corresponding successive approximations; the right-hand side of this estimate tends to zero at a rate of some geometric progression. In the case where the kernel of the equation satisfies the stochasticity condition, the absence of a nontrivial continuous bounded solution is proved. In the class of nonnegative nontrivial continuous bounded functions, a uniqueness theorem is also established. Using some geometric estimates for concave functions, the asymptotic behavior of the constructed solution at infinity is studied. At the end of the article, to illustrate the results obtained, practical examples of the kernel (subkernel) and nonlinearity of the equation under study are given.

Keywords: bounded solution, monotonicity, subkernel, concavity, successive approximations.

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