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EXTENSIBILITY OF SOLUTIONS OF NON-AUTONOMOUS SYSTEMS OF QUADRATIC DIFFERENTIAL EQUATIONS AND THEIR APPLICATION IN OPTIMAL CONTROL PROBLEMS

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The paper considers minimization problems with a free right endpoint on a given time interval for control affine systems of differential equations. For this class of problems, we study an estimate for the number of different zeros of switching functions that determine the form of the corresponding optimal controls. This study is based on the analysis of non-autonomous linear systems of differential equations for switching functions and the corresponding auxiliary functions. Non-autonomous linear systems of third order are considered in detail. In these systems, the variables are changed so that the matrix of the system is transformed to a special upper triangular form. As a result, the number of zeros of the corresponding switching functions is estimated using the generalized Rolle's theorem. In the case of a linear system of third order, this transformation is carried out using functions that satisfy a non-autonomous system of quadratic differential equations of the same order. The paper presents two approaches that ensure the extensibility of solutions to a non-autonomous system of quadratic differential equations to a given time interval. The first approach uses differential inequalities and Chaplygin's comparison theorem. The second approach combines splitting a non-autonomous system of quadratic differential equations into subsystems of lower order and applying the quasi-positivity condition to these subsystems.

Keywords: switching function, generalized Rolle's theorem, non-autonomous system of quadratic differential equations, extensibility of solutions, condition for quasi-positivity of solutions.

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