

MSC: 32E30, 41A20

DOI: 10.21538/0134-4889-2024-30-1-170-189

**A FEJÉR RATIONAL INTEGRAL OPERATOR ON A CLOSED INTERVAL  
AND APPROXIMATION OF FUNCTIONS  
WITH A POWER-LAW SINGULARITY**

**P. G. Potseiko, E. A. Rovba**

Rational approximations of continuous functions and functions with a power-law singularity on a closed interval are studied by means of integral Fejér-type operators. Upper estimates of approximations of continuous functions on a closed interval are derived; the estimates are expressed in terms of the modulus of continuity and depend on the position of a point in the interval. Rational approximations of the function  $(1-x)^\gamma$ ,  $\gamma \in (0, 1)$ , on the interval  $[-1, 1]$  are studied. Upper estimates of uniform approximations in terms of the corresponding majorant and an asymptotic expression as  $n \rightarrow \infty$  of this majorant are found. In the case of a fixed number of poles of the approximating function, optimal values of the parameters are obtained, for which the majorant of the uniform approximations decreases at the highest rate. A consequence of the results obtained is asymptotic estimates of approximations of some specific functions by Fejér sums of polynomial Fourier–Chebyshev series.

Keywords: rational approximations, Fejér integral operator, pointwise and uniform estimates of approximations, modulus of continuity, function with a power-law singularity, asymptotic estimates.

**REFERENCES**

1. Fejér L. Untersuchungen über Fouriersche Reihen. *Math. Ann.*, 1904, vol. 58, pp. 51–69. doi: 10.1007/BF01447779
2. Lebesgue H. Sur les intégrales singulières. *Annales de la faculté des sciences de Toulouse 3e série*, 1909, vol. 1, pp. 25–117.
3. Bernstein S. Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné. Bruxelles : Hayez, imprimeur des academies royales, 1912. 104 p.
4. Privalov I.I. On approximation of functions satisfying Lipschitz condition by Fejér sums. *Math. Sb.*, 1918, vol. 30, no. 4, pp. 521–526 (in Russian).
5. Nikol'skii S.M. On asymptotic behaviour of remainder in approximation of functions satisfying Lipschitz condition by Fejér sums. *Izvestiya Akad. Nauk SSSR, Ser. Matem.*, 1940, vol. 4, no. 6, pp. 501–508 (in Russian).
6. Zygmund A. On the degree of approximation of functions by Fejér means. *Bull. Amer. Math. Soc.*, 1945, vol. 51, no. 4, pp. 274–278. doi: 10.1090/S0002-9904-1945-08332-3
7. Novikov O.A., Rovenskaya O.G. Approximation of classes of Poisson integrals by Fejér sums. *Computer Research and Modeling*, 2015, vol. 7, no. 4, pp. 813–819. doi: 10.20537/2076-7633-2015-7-4-813-819
8. Efimov A. V. On approximation of some classes of continuous functions by Fourier sums and Fejér sums. *Izvestiya Akad. Nauk SSSR, Ser. Matem.*, 1958, vol. 22, no. 1, pp. 81–116 (in Russian).
9. Lebed' G.K., Avdeenko A.A. Approximation of periodic functions by Fejér sums. *Math. USSR-Izv.*, 1971, vol. 5, no. 1, pp. 86–96. doi: 10.1070/IM1971v005n01ABEH001014
10. Savchuk V.V. Approximation of functions of Dirichlet class by Fejér means. *Math. Notes*, 2007, vol. 81, no. 5, pp. 665–670. doi: 10.1134/S0001434607050124
11. Dzhrbash'an M.M. To the theory of Fourier series in rational functions. *Izvestiya Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat.*, 1956, vol. 9, no. 7, pp. 1–27 (in Russian).
12. Rusak V.N. On approximation by rational fractions. *Dokl. Akad Nauk BSSR*, 1964, vol. 8, no. 7, pp. 432–435 (in Russian).
13. Rusak V.N. On approximation by rational functions on the real line. *Izvestiya Akad. Nauk BSSR, Ser. Fiz.-Mat.*, 1974, no. 1, pp. 22–28 (in Russian).

14. Rusak V. N. *Ratsional'nye funktsii kak apparat priblizheniya* [Rational functions as approximation apparatus], Minsk, Belarus State Univ. Publ., 1979, 179 p.
15. Kitbalyan A.A. On a generalization of the Fejér kernel. *Dokl. Akad. Nauk Arm. SSR*, 1979, vol. 69, pp. 8–14 (in Russian).
16. Rovba E.A. Rational integral operators on a segment. *Vestnik Beloruss. Gos. Univ.*, 1996, vol. 1, no. 1, pp. 34–39 (in Russian).
17. Rovba E.A. On the approximation of functions of bounded variation by Fejér and Jackson rational operators. *Dokl. Nats. Akad. Nauk Belarusi*, 1998, vol. 42, no. 4, pp. 13–17 (in Russian).
18. Smotritskii K.A. On approximation of convex functions by rational integral operators on a segment. *Vestnik Beloruss. Gos. Univ.*, 2005, vol. 1, no. 3, pp. 64–70 (in Russian).
19. Bernstein S. N. *Ekstremal'nye svoistva polinomov i nailuchshee priblizhenie nepreryvnykh funktsii odnoi veshchestvennoi peremennoi* [Extremal properties of polynomials and the best approximation of continuous functions of one real variable], Part 1, Moscow–Leningrad, Ob'edinennoe Nauch. Tekh. Izd. Narod. Komiss. Tyazheloi Promyshl. SSSR, 1937, 203 p.
20. Ibragimov I.I. On asymptotic value of best approximations of functions having real singular point. *Izvestiya Akad. Nauk SSSR. Ser. Matem.*, 1946, vol. 10, no. 5, pp. 429–460 (in Russian).
21. Nikol'skii S.M. On the best approximation in the mean to the function  $|a - x|^s$  by polynomials. *Izvestiya Akad. Nauk SSSR. Ser. Matem.*, 1947, vol. 11, no. 2, pp. 139–180 (in Russian).
22. Reddy A.R. A note on rational approximation to  $(1 - x)^{1/2}$ . *J. Approx. Theory*, 1979, vol. 25, no. 1, pp. 31–33. doi: 10.1016/0021-9045(79)90030-3
23. Bundschuh P.A. Remark on Reddy's paper on the rational approximation of  $(1 - x)^{1/2}$ . *J. Approx. Theory*, 1981, vol. 32, no. 3, pp. 167–169. doi: 10.1016/0021-9045(81)90113-1
24. McD. Mercer A. A note on rational approximation to  $(1 - x)^\alpha$ . *J. Approx. Theory*, 1983, vol. 38, no. 1, pp. 101–103. doi: 10.1016/0021-9045(83)90146-6
25. Reddy A.R. A note on rational approximation to  $(1 - x)^\sigma$ . *J. Approx. Theory*, 1987, vol. 49, no. 4, pp. 404–407. doi: 10.1016/0021-9045(87)90078-5
26. Alzer H. On rational approximation of  $(1 - x)^\sigma$ . *Archiv Math.*, 1996, vol. 67, no. 2, pp. 134–137. doi: 10.1007/BF01268927
27. Andersson J.-E. Best rational approximation to Markov functions *J. Approx. Theory*, 1994, vol. 76, no. 2, pp. 219–232. doi: 10.1006/jath.1994.1015
28. Pekarskii A.A. Best uniform rational approximations to Markov functions. *St. Petersburg Math. J.*, 1996, vol. 7, no. 2, pp. 277–285.
29. Patseika P.G., Rouba Y.A., Smatrytski K.A. On one rational integral operator of Fourier–Chebyshev type and approximation of Markov functions *J. Belarusian State Univ. Math. Inform.*, 2020, vol. 2, pp. 6–27. doi: 10.33581/2520-6508-2020-2-6-27
30. Rovba E.A. One direct method in the rational approximation. *Dokl. Nats. Akad. Nauk Belarusi*, 1979, vol. 23, no. 11, pp. 968–971 (in Russian).
31. Patseika P.G., Rouba Y.A. On rational Abel–Poisson means on a segment and approximations of Markov functions. *Zhurnal Beloruss. Gos. Univ. Matem. Inform.*, 2021, vol. 3, pp. 6–24 (in Russian). doi: 10.33581/2520-6508-2021-3-6-24
32. Potseiko P.G., Rovba E.A. On rational approximations of Markov function on a segment by Fejér sums with fixed number of poles. *Trudy Inst. Mat.*, 2022, vol. 30, no. 1–2, pp. 63–83 (in Russian).
33. Misyuk V.R. On the best approximation of power function in Bergman space. In: Proc. 13th Int. Conf. Kazan Summer Scientific School–Conf. “Function theory, its applications and related questions”. Kazan: Kazan Math. Soc. publ., Acad. Sci. Resp. Tatarstan publ., 2017, vol. 54, pp. 257–259 (in Russian).
34. Timan A.F. *Theory of approximation of functions of a real variable*, Oxford, Pergamon Press, 1963, 631 p. doi: 10.1016/C2013-0-05307-8 Original Russian text was published in Timan A.F., *Teoriya priblizheniya funktsii deistvitel'nogo peremennogo*, Moscow, Gos. Izd-vo Fiz. Mat. Lit., 1960, 624 p.
35. Akhiezer N.I. *Lektsii po teorii approksimatsii* [Lectures on the Theory of Approximation], Moscow, Nauka Publ., 1965, 409 p.
36. Sidorov Yu. V., Fedoryuk M. V., Shabunin M. I. *Lektsii po teorii funktsii kompleksnogo peremennogo* [Lectures on complex variable functions theory], Moscow, Nauka Publ., Glav. Red. Fiz. Mat. Lit., 1989, 480 p.
37. Patseika P.G., Rouba Y.A. Fejér means of rational Fourier–Chebyshev series and approximation of function  $|x|^s$ . *Zhurnal Beloruss. Gos. Univ. Matem. Inform.*, 2019, vol. 3, pp. 18–34 (in Russian). doi: 10.33581/2520-6508-2019-3-18-34

Received May 15, 2023

Revised December 18, 2023

Accepted December 25, 2023

**Funding Agency:** This work was supported by the National Program for Scientific Research of the Republic of Belarus “Convergence 2020” (project no. 20162269).

*Pavel G. Potseiko*, Ph.D., Faculty of Mathematics and Informatics Yanka Kupala State University of Grodno (Belarus) Ozheshko St., 22, 230023, Grodno, Belarus; e-mail: pahamatby@gmail.com .

*Yevgeniy A. Rovba*, Dr. Phys.-Math. Sci., Prof., Faculty of Mathematics and Informatics Yanka Kupala State University of Grodno (Belarus) Ozheshko St., 22, 230023, Grodno, Belarus; e-mail: rovba.ea@gmail.com .

Cite this article as: P.G.Potseiko, E.A.Rovba. A Fejér rational integral operator on a closed interval and approximation of functions with a power-law singularity. *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2024, vol. 30, no. 1, pp. 170–189 .