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ON THE PROBLEM OF COVERING SPHERICAL FIGURES WITH EQUAL SPHERICAL CAPS

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We consider the problem of covering the surface of a three-dimensional set with a given number of elements when this set is a ball or a spherical segment and the covering elements are identical spherical caps. The optimization criterion is to minimize the radius of the spherical caps. This formulation is a relatively little-studied case of the classical circle covering problem (CCP) for a simply connected set, which is relevant in connection with applications in information and telecommunication technologies and logistics. The peculiarity of this study is that, besides the traditional Euclidean distance between points, a specific metric that characterizes the distance between points as the time of movement between them is also considered. A new heuristic algorithm is proposed, based on a spherical analog of the Voronoi diagram and the optical-geometrical analogy traditional for the authors, which allows solving the problem of covering non-planar surfaces. Since we could not find material for comparison with a general metric, the case of geodesic distance on a sphere was considered separately. For this case, we developed an algorithm for constructing the best covering based on finding the Chebyshev centers of Dirichlet zones, with a proof of the theorem that allows us to evaluate its effectiveness. Illustrative numerical calculations are performed.

Keywords: optimal covering, non-Euclidean distance, Voronoi diagram, optical-geometric analogy, Chebyshev center.

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