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QUESTIONS OF THE STRUCTURE OF FINITE HALL QUASIFIELDS

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The finite quasifields have been studied together with projective translation planes for more than a century. The identification of structural features and anomalous properties is an important step in solving the classification problem of finite quasifields. The article solves the structural problems for finite Hall quasifields. These are quasifields two-dimensional over the center such that all non-central elements are the roots of a unique quadratic equation. The automorphism group acts transitively on non-central elements. All Hall quasifields of the same order coordinatize one isomorphic translation plane, which is the Hall plane. The spread set method allows to present the multiplication rule as a linear transformation. The method is used to describe subfields, subquasifields, spectra, and automorphisms. An algorithm to calculate the number of pairwise non-isomorphic Hall quasifields of the same order is given. The covering and primitivity theorem by M. Cordero and V. Jha (2009) is clarified, with the primitive Hall quasifields counter-examples. The quasifields of order 16 covered by subfields of order 4 not contained in any Hall quasifield are presented. The examples also raise the questions for further investigation.

Keywords: quasifield, Hall quasifield, spread set, spectrum, automorphism, right-primitive quasifield.

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