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## A GENERALIZED TRANSLATION OPERATOR GENERATED BY THE SINC-FUNCTION ON AN INTERVAL

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We discuss the properties of the generalized translation operator generated by the system of functions  $\mathfrak{S} = \{(\sin k\pi x)/(k\pi x)\}_{k=1}^{\infty}$  in the spaces  $L^q = L^q((0, 1), \nu)$ ,  $q \geq 1$ , on the interval  $(0, 1)$  with the weight  $\nu(x) = x^2$ . We find an integral representation of this operator and study its norm in the spaces  $L^q$ ,  $1 \leq q \leq \infty$ . The translation operator is applied to the study of Nikol'skii's inequality between the uniform norm and the  $L^q$ -norm of polynomials in the system  $\mathfrak{S}$ .

Keywords: generalized translation, sinc-function, inequality of different metrics.

### REFERENCES

1. Watson G.N. *A treatise on the theory of Bessel functions*. Cambridge, Cambridge Univ. Press, 1995, 814 p. ISBN: 0521483913 Translated to Russian under the title *Teoriya besselevykh funktsii. Ch. 1*, Moscow, Inostr. Liter. Publ., 1949, 798 p.
2. Bateman G., Erdélyi A. *Higher transcendental functions. Vol. II*. NY, McGraw Hill Book Company, 1953, 396 p. ISBN: 0486446158 Translated to Russian under the title *Vysshie transtsendentnye funktsii. T. 2. Funktsii Besselya, funktsii parabolicheskogo tsilindra, ortogonal'nye mnogochleny*, Moscow, Nauka Publ., 1966, 295 p.
3. Vladimirov V.S. *Generalized functions in mathematical physics*. Moscow, Mir Publ., 1979, 362 p. ISBN: 071471545X. Original Russian text published in Vladimirov V.S., *Uravneniya matematicheskoi fiziki*, Moscow, Nauka Publ., 1981, 512 p.
4. Levitan B.M. Expansion in Fourier series and integrals with Bessel functions. *Uspekhi Mat. Nauk*, 1951, vol. 6, no. 2, pp. 102–143 (in Russian).
5. Babenko A.G. Exact Jackson–Stechkin inequality in the space  $L^2(\mathbb{R}^m)$ . *Trudy Inst. Mat. Mekh. UrO RAN*, 1998, vol. 5, pp. 183–198 (in Russian).
6. Platonov S.S. Bessel harmonic analysis and approximation of functions on the half-line. *Izv. Math.*, 2007, vol. 71, no. 5, pp. 1001–1048. doi: 10.1070/IM2007v071n05ABEH002379
7. Liu Y. Best  $L^2$ -approximation of function on  $[0, 1]$  with the weight  $x^{2\nu+1}$ . *Proceedings of S. B. Stechkin's International Mathematical Summer Workshop on function theory*, Tula, Tul'skii Gosudarstvennyi Universitet, 2007, pp. 180–190 (in Russian).
8. Abilov V.A., Abilova F.V., Kerimov M.K. Some issues concerning approximations of functions by Fourier–Bessel sums. *Comput. Math. Math. Phys.*, 2013, vol. 53, no. 7, pp. 867–873. doi: 10.1134/S0965542513070026
9. Arestov V., Babenko A., Deikalova M., Horváth A. Nikol'skii inequality between the uniform norm and integral norm with Bessel weight for entire functions of exponential type on the half-line. *Anal. Math.*, 2018, vol. 44, no. 1, pp. 21–42. doi: 10.1007/s10476-018-0103-6
10. Arestov V. V., Deikalova M. V. On one generalized translation and the corresponding inequality of different metrics. *Proc. Steklov Inst. Math.*, 2022, vol. 319, suppl. 1, pp. S30–S42. doi: 10.1134/S0081543822060049
11. Arestov V., Deikalova M. On one inequality of different metrics for trigonometric polynomials. *Ural Math. J.*, 2022, vol. 8, no. 2, pp. 25–43. doi: 10.15826/umj.2022.2.003
12. Levitan B.M. The application of generalized displacement operators to linear differential equations of the second order. *Uspekhi Mat. Nauk*, 1949, vol. 4, no. 1(29), pp. 3–112 (in Russian).

13. Gray A., Mathews G.B. *A treatise on Bessel functions and their applications to physics*. London, Macmillan and Co., 1895, 292 p. Translated to Russian under the title *Functsii Besselya i ikh prilozheniya k fizike i mekhanike*, Moscow, Inostr. Liter. Publ., 1953, 372 p.
14. Dunford N., Schwartz J.T. *Linear Operators. Part 1: General Theory*. NY, Interscience, 1988, 872 p. ISBN: 978-0-471-60848-6. Translated to Russian under the title *Lineinye operatory. Obshchaya teoriya*. Moscow, Editorial URSS Publ., 2004, 896 p. ISBN: 5-354-00601-5.
15. Natanson I.P. *Theory of functions of a real variable*. Dover Publ., 2016, 544 p. ISBN: 978-0486806433. Original Russian text was published in Natanson I.P. *Teoriya funktsii veshchestvennoi peremennoi*, Saint Petersburg, Lan' Publ., 1999, 560 p. ISBN: 5-8114-0136-1.
16. Stein E.M., Weiss G. *Introduction to Fourier analysis on euclidean spaces*. Princeton, Princeton Univ. Press, 1971, 312 p. ISBN: 9780691080789. Translated to Russian under the title *Vvedenie v garmonicheskii analiz na evklidovykh prostranstvakh*, Moscow, Mir Publ., 1974, 333 p.
17. Jackson D. Certain problems of closest approximation. *Bull. Amer. Math. Soc.*, 1933, vol. 39, no. 12, pp. 889–906. doi: 10.1090/S0002-9904-1933-05759-2
18. Nikol'skii S.M. Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of several variables. *Trudy Mat. Inst. Steklov*, 1951, vol. 38, pp. 244–278 (in Russian).
19. Nikol'skii S.M. *Approximation of functions of several variables and imbedding theorems*. NY, Springer, 1975, 420 p. doi: 10.1007/978-3-642-65711-5 Original Russian text published under the title *Priblizhenie funktsii mnogikh peremennykh i teoremy vlozheniya*, Moscow, Nauka Publ., 1969, 480 p.
20. Bernstein S.N. *Ekstremal'nye svoystva polinomov i nailuchshee priblizhenie nepreryvnykh funktsii odnoi veshchestvennoi peremennoi* [Extremal properties of polynomials and the best approximation of continuous functions of one real variable], Part 1. Moscow, Leningrad, Ob'edinennoe Nauch. Tekh. Izd. Narod. Komiss. Tyazheloi Promyshl. SSSR, 1937, 203 p.
21. Korneichuk N.P., Babenko V.F., Ligon A.A. *Extremal properties of polynomials and splines*. NY, Nova Science Publ., 1996, 439 p. ISBN: 978-1560723615. Original Russian text was published in Korneichuk N.P., Babenko V.F., Ligon A.A., *Ekstremal'nye svoystva polinomov i splainov*, Kiev, Naukova Dumka Publ., 1992, 304 p. ISBN: 5-12-002210-3.
22. Milovanović G.V., Mitrinović D.S., Rassias Th.M. *Topics in polynomials: extremal problems, inequalities, zeros*. Singapore, World Scientific, 1994, 821 p. ISBN: 981-02-0499-X.
23. Borwein P., Erdélyi T. *Polynomials and polynomial inequalities*. NY, Springer-Verlag, 1995, 480 p. doi: 10.1007/978-1-4612-0793-1
24. Rahman Q.I., Schmeisser G. *Analytic theory of polynomials*. Oxford, Oxford Univ. Press, 2002, 742 p. ISBN: 0-19-853493-0.
25. Babenko V.F., Korneichuk N.P., Kofanov V.A., Pichugov S.A. *Neravenstva dlya proizvodnykh i ikh prilozheniya* [Inequalities for derivatives and their applications]. Kiev, Naukova Dumka Publ., 2003, 590 p.
26. Bari N.K. Generalization of inequalities of S. N. Bernshtein and A. A. Markov. *Izv. Akad. Nauk SSSR Ser. Mat.*, 1954, vol. 18, no. 2, pp. 159–176 (in Russian).
27. Ivanov V.I. Certain inequalities in various metrics for trigonometric polynomials and their derivatives. *Math. Notes*, 1975, vol. 18, no. 4, pp. 880–885. doi: 10.1007/BF01153038
28. Arestov V.V. Inequality of different metrics for trigonometric polynomials. *Math. Notes*, 1980, vol. 27, no. 4, pp. 265–269. doi: 10.1007/BF01140526
29. Badkov V.M. Asymptotic and extremal properties of orthogonal polynomials corresponding to weight having singularities. *Proc. Steklov Inst. Math.*, 1994, vol. 198, pp. 37–82.
30. Babenko V., Kofanov V., Pichugov S. *Comparison of rearrangement and Kolmogorov–Nagy type inequalities for periodic functions*. In: Approx. Theory: A volume dedicated to Blagovest Sendov, ed. B. Bojanov, Sofia, DARBA, 2002, pp. 24–53.
31. Gorbachev D.V. Sharp Bernstein–Nikolskii inequalities for polynomials and entire functions of exponential type. *Chebyshevskii Sbornik*, 2021, vol. 22, no. 5, pp. 58–110 (in Russian). doi: 10.22405/2226-8383-2021-22-5-58-110
32. Arestov V.V., Deikalova M.V. Nikol'skii inequality for algebraic polynomials on a multidimensional Euclidean sphere. *Proc. Steklov Inst. Math. Suppl.*, 2014, vol. 284, suppl. 1, pp. 9–23. doi: 10.1134/S0081543814020023
33. Arestov V., Deikalova M. Nikol'skii inequality between the uniform norm and  $L_q$ -norm with ultraspherical weight of algebraic polynomials on an interval. *Comput. Methods Funct. Theory*, 2015, vol. 15, no. 4, pp. 689–708. doi: 10.1007/s40315-015-0134-y

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34. Arestov V., Deikalova M. Nikol'skii inequality between the uniform norm and  $L_q$ -norm with Jacobi weight of algebraic polynomials on an interval. *Analysis Math.*, 2016, vol. 42, no. 2. pp. 91–120. doi: 10.1007/s10476-016-0201-2
  35. Arestov V., Deikalova M., Horváth Á. On Nikol'skii type inequality between the uniform norm and the integral  $q$ -norm with Laguerre weight of algebraic polynomials on the half-line. *J. Approx. Theory*, 2017, vol. 222, pp. 40–54. doi: 10.1016/j.jat.2017.05.005
  36. Arestov V.V. A characterization of extremal elements in some linear problems. *Ural Math. J.*, 2017, vol. 3, no. 2, pp. 22–32. doi: 10.15826/umj.2017.2.004

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