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ON CONSTANTS IN THE BERNSTEIN–SZEGŐ INEQUALITY FOR THE WEYL DERIVATIVE OF ORDER LESS THAN UNITY OF TRIGONOMETRIC POLYNOMIALS AND ENTIRE FUNCTIONS OF EXPONENTIAL TYPE IN THE UNIFORM NORM

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In the set $\mathscr{T}_n$ of trigonometric polynomials f_n of order n with complex coefficients, the Weyl derivative (fractional derivative) $f_n^{(\alpha)}$ of real nonnegative order α is considered. We study the question about the constant in the Bernstein–Szegő inequality $\left\|f_n^{(\alpha)}\cos\theta+\tilde{f}_n^{(\alpha)}\sin\theta\right\|\leqslant B_n(\alpha,\theta)\|f_n\|$ in the uniform norm. This inequality has been well studied for $\alpha\geqslant 1$: G. T. Sokolov proved in 1935 that it holds with the constant n^α for all $\theta\in\mathbb{R}$. For $0<\alpha<1$, there is much less information about $B_n(\alpha,\theta)$. In this paper, for $0<\alpha<1$ and $\theta\in\mathbb{R}$, we establish the limit relation $\lim_{n\to\infty}B_n(\alpha,\theta)/n^\alpha=\mathcal{B}(\alpha,\theta)$, where $\mathcal{B}(\alpha,\theta)$ is the sharp constant in the similar inequality for entire functions of exponential type at most 1 that are bounded on the real line. The value $\theta=-\pi\alpha/2$ corresponds to the Riesz derivative, which is an important particular case of the Weyl–Szegő operator. In this case, we derive an exact asymptotic expansion for the quantity $B_n(\alpha)=B_n(\alpha,-\pi\alpha/2)$ as $n\to\infty$.

Keywords: trigonometric polynomials, entire functions of exponential type, Weyl–Szegő operator, Riesz derivative, Bernstein inequality, uniform norm.

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