POLYNOMIAL APPROXIMABILITY OF THE ASYMMETRIC PROBLEM OF COVERING A GRAPH BY A BOUNDED NUMBER OF CYCLES

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Recently, O. Svensson and V. Traub provided the first proof of the polynomial-time approximability with fixed ratios for the Asymmetric Traveling Salesman Problem (ATSP). Just as the famous Christofides–Serdyukov algorithm for the symmetric routing problems, this breakthrough result, applied as a "black box," has opened an opportunity for developing the first polynomial-time approximation algorithms with constant ratios for several related combinatorial problems. At the same time, problems have been revealed in which this simple approach, based on reducing a given instance to one or more auxiliary ATSP instances, does not succeed. In the present paper, we extend the Svensson–Traub approach to the wider class of problems about finding a minimum-weight cycle cover of an edge-weighted directed graph with an additional constraint on the number of cycles. In particular, it is shown for the first time that the Minimum Weight Cycle Cover Problem with at most k cycles admits a polynomial-time approximation with constant ratio $\max\{22 + \varepsilon, 4 + k\}$ for arbitrary $\varepsilon > 0$.

Keywords: cycle cover of a graph, asymmetric traveling salesman problem, fixed ratio approximation algorithm.

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