THE PROBLEM OF DIFFUSION WAVE INITIATION FOR A NONLINEAR SECOND-ORDER PARABOLIC SYSTEM

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The study of nonlinear singular parabolic equations occupies a key place in the scientific school of A. F. Sidorov. In particular, the problem on initiating a heat wave has been studied since the 1980s. The present study aims to extend the results of Sidorov and his followers, including the authors, to the case of systems of the corresponding type. We find that the heat (diffusion) wave for the system considered has a more complex (three-part) structure, which follows from the fact that the zero fronts are different for the unknown functions. A theorem on the existence and uniqueness of a piecewise analytical solution, which has the form of special series, is proved. We find an exact solution of the desired type, the construction of which is reduced to the integration of ordinary differential equations (ODEs). We managed to integrate the ODEs by quadratures. In addition, we propose an algorithm based on the collocation method, which allows us to effectively construct an approximate solution on a given time interval. Illustrative numerical calculations are performed. Since we have not managed to prove the convergence in this case (this is far from always possible for nonlinear singular equations and systems), exact solutions, both obtained in this paper and previously known, have been used to verify the calculation results.

Keywords: nonlinear parabolic system, singularity, existence theorem, special series, exact solution, collocation method, computational experiment.

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