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ON THE DEVELOPMENT OF THE VARIATIONAL APPROACH TO THE GENERATION OF OPTIMAL GRIDS (A SURVEY)

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A survey of the more than a half-century development of the variational approach to the generation of optimal grids suggested by A. F. Sidorov is presented in the paper. The idea of the approach is based on the requirements that the grid is close to a uniform orthogonal grid and is adjusted to a given function or to the solution of partial differential equations; these requirements are chosen as optimality criteria. The implementation of this idea for the generation of structured grids in two- and three-dimensional domains of geometrically complex shape is given. The developed grid generation algorithms and their applications are described. The survey is divided into two periods: the years of Sidorov's life and the subsequent years. The constructions of the functionals that formalize the grid optimality criteria are presented in relation to a unified technology created in the second period for the numerical simulation of vortex processes in multicomponent hydrodynamics. Examples of grid calculations are given using the currently developed grid generation algorithm in volumes obtained by deformations of volumes of revolution by generalizations of volumes of revolution. A volume of revolution is understood as a shape formed by the rotation of a plane generatrix consisting of segments of straight lines, arcs of circles, and ellipses, called elements, by 180° around an axis. A generalization of a volume of revolution is a volume formed by surfaces obtained by rotating elements of plane generatrices by 180° about parallel axes. A deformed volume of revolution is a volume obtained by deforming a volume of revolution by another volume of revolution or by a generalization of the volume of revolution. The cases of volumes of revolution, generalizations of volumes of revolution, and volumes of revolution deformed by volumes of revolution have formed the described grid generation technology. A basic structure in the technology is a volume of revolution, which made it possible to carry out its further development in the direction of complication of shapes of domains. At present, it is possible to build structured grids in very complicated three-dimensional domains. This possibility appeared due to the application of the moving grid technique, which is naturally implemented in variational constructions, and also due to the development of a nonstationary algorithm that deforms a volume of revolution up to a desired deformed shape and deforms and optimizes the grid in order to satisfy the optimality criteria.

Keywords: structured grids, optimal grids, moving grids, generation of grids in deformed volumes.

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