Trudy Instituta Matematiki i Mekhaniki UrO RAN

Vol. 29 No. 2 2023

MSC: 35C05, 35C99, 35N99, 35Q35, 76R10

DOI: 10.21538/0134-4889-2023-29-2-189-206

ON SOME CLASSES OF FREE CONVECTION MOTIONS

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A system of equations of unsteady spatial free convection of an incompressible viscous fluid in the Boussinesq approximation is considered. The analysis is based on the methods of reduction of linear and nonlinear partial differential equations (PDEs) and systems of PDEs to ordinary differential equations (ODEs) and systems of ODEs. These methods were proposed by the authors earlier, and their general principles are given in the paper. The methods are based on the construction of a system of equations of characteristics for a first-order PDE (the basic equation). This equation is constructed in a certain way by analyzing the original system of equations. The reductions lead to ODEs or systems of ODEs in which an independent variable ψ is such that the equation $\psi(x, y, z, t) = \text{const}$ defines a level surface for all unknown functions of the original system of PDEs. The methods are applicable to PDEs and systems of PDEs regardless of their type. The Oberbeck–Boussinesq equations are reduced to a system of ODEs with a functional arbitrariness, and an exact solution with a constant arbitrariness is found for the original system. The functional arbitrariness in the constructed reduction also yielded a system of ODEs in which the temperature T is an independent variable. For this system exact solutions are found. A possible (vortex or vortex-free) motion of an incompressible fluid with free convection is analyzed. The cases of vortex and vortex-free motion of the fluid are identified. An exact solution defining a vortex-free motion of the fluid is written as a result of reductions for the original system of PDEs.

Keywords: free convection of viscous fluid, Oberbeck–Boussinesq equations, partial differential equations, reductions, exact solutions.

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Received March 7, 2023
Revised April 24, 2023
Accepted May 15, 2023

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Cite this article as: O. N. Ul'yanov, L. I. Rubina. On some classes of free convection motions. *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2023, vol. 29, no. 2, pp. 189–206 .