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EXACT CONSTANTS IN JACKSON–STECHKIN INEQUALITY IN L^2 WITH A POWER-LAW WEIGHT

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In this paper, we have solved several extremal problems of the best mean-square approximation of function f, on the semiaxis with a power-law weight, which can be used to solve various problems. Sharp Jackson–Stechkin type inequalities are obtained on some classes of functions in which the values of the best approximations are estimated from above through moduli of smoothness of the k-th order.

Keywords: exact constants in Jackson–Stechkin inequality, moduli of smoothness, best approximations, Bessel function.

Т. Е. Тилеубаев. Точные константы в неравенстве Джексона — Стечкина в L^2 со степенным весом.

В статье решено несколько экстремальных задач наилучшего среднеквадратичного приближения функции f на полуоси со степенным весом. Полученные результаты можно использовать для решения различных задач. Доказаны точные неравенства типа Джексона — Стечкина на некоторых классах функций, в которых значения наилучших приближений оцениваются сверху через модули гладкости k-го порядка.

Ключевые слова: точные константы в неравенстве Джексона — Стечкина, модули гладкости, наилучшие приближения, функция Бесселя.

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1. Introduction

The problem of finding exact constants in Jackson–Stechkin inequalities

$$E_{n-1}(f) \leq \Xi^* \omega_k(f, \frac{\gamma}{n}), \quad f \in L^2,$$

is one of the most important problems in the theory of approximations in L^2.

Formulation of the problem. It is required to determine the value of the following quantity

$$\Xi^* = \Xi^*(n, k, \gamma) = \sup \Big\{ \frac{E_{n-1}(f)}{\omega_k(f, \frac{\gamma}{n})} \colon f \in L^2 \Big\}$$

for fixed natural numbers n, k and the real number $\gamma > 0$. Here $E_{n-1}(f)$ is the best approximation of the function $f \in L^2$ by trigonometric polynomials of order $\leq n-1$, $\omega_k(f, \dfrac{\gamma}{n})$ is the modulus of smoothness of the k-th order of the function $f \in L^2$. The first exact constants in Jackson inequalities (with the first modulus of continuity) were obtained by N. P. Korneichuk [14] in space $C[0, 2\pi]$, N. I. Chernykh in $L^2[0, 2\pi]$. In 1967 N.I. Chernykh proved the following theorem:

Theorem A [6, Theorem]. *Let $f \in L^2[0, 2\pi]$, f is not equivalent to zero. Then for any $n = 0, 1, 2, \ldots$ the inequality holds*

$$E_n(f) < \frac{1}{\sqrt{2}} \omega_1(\frac{\pi}{n+1}, f).$$

Also, as a consequence of the theorem 1 from [7], he obtained the following result:

Theorem B [7, Corollary 1]. *Let r and n be natural numbers. Then for any function f for which $f^{(r)} \in L_2[0, 2\pi]$ the inequality holds*

$$E_n(f) \le \frac{1}{\sqrt{2}n^r}\Big(\frac{n}{2}\int\limits_0^{\pi/n} \omega_1^2(f^{(r)}, t)\sin nt\, dt\Big)^{1/2}.$$

In both theorems A and B, the inequalities are exact (unimproveable).

In 1979 was obtained by L. V. Taikov in [28] an exact result. In 1991 generalization of the result of N. I. Chernykh was obtained by V. V. Shalaev [25]. The generalization and development of the topic was reflected in the studies of I. I. Ibragimov and F. G. Nasibov [12] and A. Yu. Popov [21]. They extended the above result for the square summable function on the entire axis. The exact Jackson inequalities on the sphere were obtained by V. V. Arestov and V. Yu. Popov in [2]. In 1998 A. G. Babenko [3] extended the result of N. I. Chernykh for the function summable with a square on the semiaxis $\mathbb{R}_+$ with weight $t^{2\alpha+1}$, namely

Theorem C [3, Theorem 1]. *Let $\sigma > 0$, $\alpha \ge -\frac{1}{2}$. Then for any function $f \in L^2(\mathbb{R}_+, t^{2\alpha+1})$, f is not equivalent to zero, the inequalities holds*

$$E_\sigma(f) < \omega_k\Big(f, \frac{\tau}{\sigma}\Big) \quad for \quad k \ge 1, \quad \tau \ge 2q_{\alpha,1};$$

$$E_\sigma(f) < 2^{\frac{(1-k)}{2}}\omega_k\Big(f, \frac{\tau}{\sigma}\Big) \quad for \quad 0 < k < 1, \quad \tau > 2q_{\alpha,1};$$

where $q_{\alpha,1}$ is the first positive zero of the Bessel function J_α of order α.

In Theorem C, the first inequality is exact. The issue of accuracy of the second inequality remains open. Note that for $k = 1$ the assertions of Theorem C were independently obtained by A. V. Moskovsky [17].

Further development and dissemination of the topic was reflected in the studies of E. Berdisheva [4], A. V. Moskovsky [17; 18] and D. V. Gorbachev [9] in the multidimensional case. These results were preceded by a fundamental result of V. A. Yudin [35] on the exact Jackson inequality in the space L^2 on a multidimensional torus.

The extension of this question to the case of the best mean-square approximation by entire functions of exponential $\sigma > 0$ type in $L_2[(0, \infty), x^\gamma]$ with weight was carried out by V. I. Ivanov [13], A. G. Vakarchuk [31], A. G. Babenko [3], M. Sh. Shabozov [24], K. Tukhliev [22], V. V. Arestov, A. G. Babenko, M. V. Deikalova, and A. Horv'ath [1] and see the literature cited there. In the case of approximation of 2π-periodic function from $L^2[0, 2\pi]$ by the subspace of trigonometric polynomials of order $\le n-1$ in the metric $L^2[0, 2\pi]$, similar problems were solved by M. G. Esmaganbetov [8] and also in [6; 7; 25; 26; 36], the literature cited there.

This article is devoted to obtaining exact constants

$$K_{m,n,r}(t) = \sup\Big\{\frac{n^r E_{n-1}(f)}{\omega_m(B^r f, t)} : f \in W^r_{2,\mu_\alpha}(B)\Big\}$$

in Jackson–Stechkin type inequalities

$$E_{n-1}(f) \le K n^{-r}\omega_m(B^r f, t), \quad f \in W^r_{2,\mu_\alpha}(B).$$

Definitions $B^r f$ and $W^r_{2,\mu_\alpha}(B)$ are given in Section 2.

2. Basic notation and auxiliary results

2.1. Basic notation

Let $\alpha > -\dfrac{1}{2}$. By L_{2,μ_α} we denote the space consisting of measurable functions f in $[0,\infty)$ with the norm

$$\|f\| = \|f\|_{2,\mu_\alpha} = \Big(\int\limits_0^\infty |f(x)|^2 d\mu_\alpha(x)\Big)^{1/2},$$

where $d\mu_\alpha(x) = \dfrac{x^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}\,dx$.

The Hankel transform is the following integral transformation [5; 11]:

$$\mathcal{H}\colon f(t) \mapsto \hat{H}_\alpha(f)(\lambda) = \int\limits_0^\infty f(t) j_\alpha(\lambda t) d\mu_\alpha(t), \quad \lambda \in \mathbb{R}.$$

The inverse Hankel transform is given by the formula

$$\mathcal{H}^{-1}\colon f(\lambda) \mapsto \tilde{f}(t) = \tilde{H}_\alpha(f)(t) = \int\limits_0^\infty \hat{H}_\alpha(f)(\lambda) j_\alpha(\lambda t) d\mu_\alpha(\lambda).$$

Let $T > 0$ and denote by $S_T(f,x)$ the partial Hankel integral of a function $f \in L_{2,\mu_\alpha}$, i.e.,

$$S_T(f,x) = \int\limits_0^T \hat{H}_\alpha(f)(\lambda) j_\alpha(\lambda x) d\mu_\alpha(\lambda), \quad x \in (0,\infty).$$

For functions $f, g \in L_{2,\mu_\alpha}$, the generalized Plancherel's theorem [29; 30] $\big\langle f, g\big\rangle = \big\langle \hat{H}_\alpha f, \hat{H}_\alpha g\big\rangle$ where $\langle f, g\rangle = \displaystyle\int_0^\infty f(x)\overline{g(x)} d\mu_\alpha$ is the inner product of f and g.

We denote by $M(\sigma, 2, \alpha)$, $\sigma > 0$, the set of all functions $Q_\sigma(x)$ satisfying the following conditions (see [20]):

1. $Q_\sigma(x)$ — even, entire function of exponential type σ;
2. $Q_\sigma(x)$ belong to class L_{2,μ_α}.

The best approximation of a function $f \in L_{2,\mu_\alpha}$ by the class $M(\sigma, p, \alpha)$, $p = 2$, $\sigma > 0$ is defined as follows:

$$E_\sigma(f) = E_\sigma(f)_{2,\mu_\alpha} = \inf\{\|f - Q_\sigma\| \colon Q_\sigma \in M(\sigma, 2, \alpha)\} = \Big(\int\limits_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\Big)^{1/2}. \qquad (2.1)$$

We denote by $j_\alpha(\lambda t)$ the normalized Bessel function [30; 34] $j_\alpha(\lambda t) = \dfrac{2^\alpha\Gamma(\alpha+1)J_\alpha(\lambda t)}{(\lambda t)^\alpha}$. We recall that for all $\lambda \in \mathbb{C}$, the function $j_\alpha(\lambda t)$ is unique solution of the problem [15]

$$\begin{cases} Bu(t) = -\lambda^2 u(t), \\ u(0) = 1, u'(0) = 0, \end{cases}$$

where B is a differential Bessel operator on $[0,\infty)$ defined by $Bu(t) = \dfrac{d^2u}{dt^2} + \dfrac{2\alpha+1}{t}\dfrac{du}{dt}$.

We introduce a class of functions. Let S be the space of test functions on $\mathbb{R}$, i.e., the set of all infinitely differentiable functions $\varphi(t)$ decreasing as $|t| \mapsto \infty$ together with all derivatives faster than any power $|t|^{-1}$. In the usual way, the space S is endowed with a topology and becomes a locally convex space (see [30] or [20]). Let $S^{'}$ be the set of linear continuous functionals on S, i.e., the space of generalized functions of slow growth. In the usual way, $S^{'}$ is endowed with the structure of a topological vector space. Denote by S_+ be the subspace in S consisting of even functions. The space S_+ is endowed with the induced from the space S topology.

Let $S_+^{'}$ be the space of even distributions of slow growth, i.e., the set of continuous linear functionals on S_+. For $f \in S_+^{'}$ and $\varphi \in S_+$, we denote by (f, φ) the value of the functional f on the function φ.

The spaces L_{p,μ_α}, $1 \leq p \leq \infty$, are embedded in the space $S_+^{'}$ if for $f(t) \in L_{p,\mu_\alpha}$ and $\varphi(t) \in S_+$ we set

$$(f, \varphi) = \int_0^\infty f(t)\varphi(t)t^{2\alpha+1}dt.$$

Since the Bessel differential operator satisfies the condition

$$(B\varphi, \phi) = (\varphi, B\phi), \quad \phi,\ \varphi \in S_+,$$

then the action of the operator B naturally extends to generalized functions by the formula

$$(Bf, \varphi) = (f, B\varphi), \quad f \in S_+^{'}, \quad \varphi \in S_+.$$

It is known that subspace S_+ is dense in space L_{p,μ_α} (see [20;30]). Since S_+ is a dense linear subset in a Banach space $L_{p,\mu_\alpha}$for $p < \infty$, it follows from inequality

$$\|T_h f\|_{p,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha}$$

that the operator T_h on S_+ extends by continuity to a bounded operator T_h in L_{p,μ_α}, where T_h is the generalized shift operator with step $h \in [0, \infty)$ in L_{p,μ_α} (see [15;3;20] and formula (2.2) below).

If $g(t)$ is any even infinitely differentiable function, any derivative of which grows as $t \to \infty$ no faster than some power of t (we will call such functions infinitely differentiable functions of polynomial growth), then the product of the function $g(t)$ and the generalized function $f \in S_+^{'}$. By definition, $(gf, \varphi) := (f, g\varphi)$, $\varphi \in S_+$.

For any function $f(x) \in S_+$ and for any positive numbers $r > 0$, $\sigma > 0$ we put $D_\sigma^r f = \hat{H}_\alpha(\varphi_\sigma(t)\tilde{f}(t))$, where

$$\varphi_\sigma(t) = \sum_{k=0}^{\infty} c_k^\sigma j_\alpha\big(\frac{\pi}{\sigma}kt\big)$$

is continuous, even, bounded on $\mathbb{R}$, and coincides with the function x^{2r} for $|x| \leq \sigma$.

Since $\tilde{f}(t) \in S_+$ and $\varphi_\sigma(t) \in S_+^{'}$, then $\varphi_\sigma(t)\tilde{f}(t) \in S_+^{'}$ and, therefore, $D_\sigma^r f \in S_+^{'}$.

Theorem D [20, Theorem 3.3 (Platonov)]. *If $f \in S_+$, $r > 0$, $\sigma > 0$, then the function $D_\sigma^r f$ belongs to the space L_{p,μ_α} for any $1 \leq p \leq \infty$ and the inequality holds*

$$\|D_\sigma^r f\|_{p,\mu_\alpha} \leq C\sigma^{2r}\|f\|_{p,\mu_\alpha}$$

where $C = C(\alpha, r) > 0$ is the some constant.

We assume by definition

$$(-B)^r f = D_\sigma^r f.$$

In Lemma 3.5 by Platonov [20] proves the correctness of the definition of the operator $(-B)^r$. In addition, in [20, Theorem 1.1] an analogue of the direct approximation theorem in the space L_{p,μ_α}, $1 \leq p \leq \infty$, $\alpha > -1/2$, is established.

Let $W^r_{2,\mu_\alpha}(B), r = 1, 2, \dots$ is a Sobolev space(see [20]), constructed by the differential operator B, i.e.

$$W^r_{2,\mu_\alpha}(B) = \left\{f \in L_{2,\mu_\alpha} : B^j f \in L_{2,\mu_\alpha}, j = 1, 2, \dots, r\right\}.$$

Consider in the space $L_{p,\mu_\alpha}$the generalized shift operator of functions $f(x)$ (see [15;3;20]

$$(T_h f)(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int\limits_0^\pi f(\sqrt{x^2+h^2-2xh\cos\varphi})(\sin\varphi)^{2\alpha} d\varphi. \tag{2.2}$$

In [3] the generalized difference operator and the moduli of smoothness of the m-th order was determined as follows:

$$\delta_t^m f = (I - T_t)^{m/2} f = \sum_{k=0}^\infty (-1)^k \binom{\frac{m}{2}}{k} T_t^k f$$

and

$$\omega_m(f,t) = \sup_{0\le h\le t} \|\delta_h^m f\|_{2,\mu_\alpha} = \sup_{0<h\le t} \left\{ \int\limits_0^\infty (1 - j_\alpha(\lambda h))^m |\hat{f}_\alpha(\lambda)|^2 d\mu_\alpha(\lambda) \right\}^{1/2}, \tag{2.3}$$

where I is the identity operator and $T_t^0 = I$.

In [3], when solving problems of the theory approximations in the space $L_2(\mathbb{R}^d)$ associated with finding the exact constants in the Jackson–Stechkin inequality

$$E_\sigma(f)_{L_2(\mathbb{R}^d)} \le \omega_m\big(f, \frac{\tau}{\sigma}\big)_{L_2(\mathbb{R}^d)}$$

considered the following extreme characteristic:

$$K_{\sigma,m,d} = \sup\left\{ \frac{E_\sigma(f)_{L_2(\mathbb{R}^d)}}{\omega_m\big(f, \frac{\tau}{\sigma}\big)_{L_2(\mathbb{R}^d)}} : f \in L_2(\mathbb{R}^d) \right\}$$

and found the exact constant. More detailed information and corresponding notations are given below in Section 4. Note that the notation $A_\sigma f$ (see (4.2)) is used there instead of $E_\sigma(f)_{L_2(\mathbb{R}^d)}$.

In this paper, we will consider solving approximation problems in L_{2,μ_α} associated with finding exact constants in Jackson's inequality

$$E_\sigma(f) \le K\sigma^{-2r}\omega_r\big(B^r f, \frac{\tau}{\sigma}\big)$$

for functions $f \in W^r_{2,\mu_\alpha}(B)$. The finiteness of the exact constant follows from the above-mentioned result of Platonov [20, Theorem 1.1]. Also, in this paper, we consider an extreme approximate characteristic of the following form

$$\Xi_{\sigma,r,m,p}(h) = \sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r} E_\sigma(f)}{\Big(\int_0^h \omega_m^p(B^r f, t)\varphi(t)dt\Big)^{1/p}}, \tag{2.4}$$

where $r \in \mathbb{Z}_+$, $m \in \mathbb{N}$, $0 < p < 2$, $\sigma > 0$, $h \in \big[0, \frac{q_{\alpha+1,1}}{\sigma}\big]$, $q_{\alpha+1,1}$ is the smallest positive zero of the function $j_{\alpha+1}(t)$, $0 \le t < \infty$, and $\varphi(t)$ is a non-negative, measurable, summable on the interval $[0, h]$ and not equivalent to zero function.

2.2. Auxiliary results

Let $q_{\alpha,l}$ be the positive l-th root of the Bessel functions $J_\alpha(x)$, i.e.

$$J_\alpha(q_{\alpha,l}) = 0, \quad l \in \mathbb{N}, \quad 0 < q_{\alpha,1} < q_{\alpha,2} < \cdots < q_{\alpha,l} < \cdots .$$

Obviously, $j_\alpha(x)$ has the same positive roots as $J_\alpha(x)$ ([34, Ch. 15, p. 526]), $j_\alpha(x)$ is continuous on the semiaxis $[0,\infty)$. In the rest of the article, we denote the segment $[q_{\alpha,k}, q_{\alpha,k+1}]$ by $I_k^{(\alpha)}$ with $k \in \mathbb{N}$, and the segment $[0, q_{\alpha,1}]$ through $I_0^{(\alpha)}$.

Since $j_\alpha(x) = 1 + O(x^2)$ for $x \longrightarrow 0$ all positive roots of the function $j_\alpha(x)$ are different, we can get that

$$j_\alpha(x) \ge 0, \quad x \in I_{2k}^{(\alpha)}, \quad \text{at} \quad k \in \mathbb{N} \cup \{0\}, \tag{2.5}$$
$$j_\alpha(x) \le 0, \quad x \in I_{2k-1}^{(\alpha)}, \quad \text{at} \quad k \in \mathbb{N}.$$

It follows from ([34, Ch. 15, p. 528]) that the positive roots of $J_\alpha(x)$ alternate with the roots of $J_{\alpha+1}(x)$, i.e.

$$0 < q_{\alpha,1} < q_{\alpha+1,1} < q_{\alpha,2} < q_{\alpha+1,2} < \cdots < q_{\alpha,l} < q_{\alpha+1,l} < \cdots . \tag{2.6}$$

To prove the theorems given in Section 3, we need the following lemmas.

Lemma 1. *Let $q_{\alpha+1,1}$ is the smallest positive zero of the function $j_{\alpha+1}(t)$. Let $\sigma > 0$ and $t \in \big(0, \frac{q_{\alpha+1,1}}{\sigma}\big]$. Then*

$$\sup_{0 \le h \le t} (1 - j_\alpha(\sigma h)) = (1 - j_\alpha(\sigma t)).$$

Proof of Lemma 1. Since

$$j_\alpha'(t) = 2^\alpha \Gamma(\alpha+1) \frac{d}{dt}\Big(\frac{J_\alpha(t)}{t^\alpha}\Big) = -\frac{t}{2(\alpha+1)} j_{\alpha+1}(t), \quad t \in I_0^{(\alpha+1)}$$

(см. [34]), then taking into account that $j_{\alpha+1}(0) > 0$ and $j_{\alpha+1}(q_{\alpha+1,1}) = 0$ from the inequalities (2.5) and (2.6), we have $j_{\alpha+1}(t) \ge 0$ for all $t \in I_0^{(\alpha+1)}$. Whence it follows that

$$(1 - j_\alpha(t))' = \frac{t}{2(\alpha+1)} j_{\alpha+1}(t) > 0 \quad \text{on} \quad I_0^{(\alpha+1)}.$$

Then the function $1 - j_\alpha(t)$ is increasing on $I_0^{(\alpha+1)}$. Therefore, for all $t \in (0, q_{\alpha+1,1}]$ we have $\sup\limits_{0<h\le t} (1 - j_\alpha(h)) = 1 - j_\alpha(t)$. Therefore, for all $t \in \big(0, \frac{q_{\alpha+1,1}}{\sigma}\big]$ we have

$$\sup_{0<h\le t} (1 - j_\alpha(\sigma h)) = 1 - j_\alpha(\sigma t).$$

Lemma is proved.

Lemma 2. *Let $k \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $q_{\alpha+1,1}$ is the smallest positive zero of the function $j_{\alpha+1}(t)$, $h \in \big[0, \frac{q_{\alpha+1,1}}{\sigma}\big]$ and $\sigma > 0$. Let*

$$\Psi(y) = y^{4r} \int_0^h (1 - j_\alpha(yt))^{2k} dt, \quad y \in G, \quad \textit{where} \quad G = \{y \colon \sigma \le y < \infty\}.$$

Then

$$\min\{\Psi(y) \colon y \in G\} = \sigma^{4r} \int_0^h (1 - j_\alpha(\sigma t))^{2k} dt.$$

Proof of Lemma 2. Since $j'_\alpha(t) = -\dfrac{t}{2(\alpha+1)} j_{\alpha+1}(t)$, $0 \le t \le \infty$, then

$$\Psi'(y) = 4ry^{4r-1}\int\limits_0^h (1-j_\alpha(yt))^{2k}dt + y^{4r}\int\limits_0^h \frac{\partial}{\partial y}\left((1-j_\alpha(yt))^{2k}\right)dt. \tag{2.7}$$

Since it is not difficult to verify, by direct verification, the equality is true

$$\frac{\partial}{\partial y}\left((1-j_\alpha(yt))^{2k}\right) = \frac{t}{y}\frac{\partial}{\partial t}\left((1-j_\alpha(yt))^{2k}\right), \tag{2.8}$$

where t, y are nonzero, then from (2.7), by virtue of (2.8), we have

$$\Psi'(y) = y^{4r-1}\left[4r\int\limits_0^h (1-j_\alpha(yt))^{2k}dt + \int\limits_0^h t\frac{\partial}{\partial t}\left((1-j_\alpha(yt))^{2k}\right)dt\right]. \tag{2.9}$$

Applying the method of integration by parts when calculating the second integral on the right side (2.9), we arrive at the following conclusion

$$\Psi'(y) = y^{4r-1}\left[(4r-1)\int\limits_0^h (1-j_\alpha(yt))^{2k}dt + h\left(1-j_\alpha(yh)\right)^{2k}\right]. \tag{2.10}$$

Since $|j_\alpha(u)| \le 1$ $\forall u \ge 0$ (see [3, formula (21)]), by virtue of (2.10) we have $\Psi'(y) > 0$ for all $y \ge \sigma$.
Lemma is proved.

Remark 1. Earlier, Lemma 2 was proved by Taikov in [28] for $\alpha = -1/2$, $\Psi(y) = y^{2r}\int_0^h (1-\cos yt)^m dt$, (and also see [8] in the more general case of Ψ), and in [33, p. 106; 26, p. 320] for $\alpha = 1/2$.

3. Main results

Let $\alpha > -1/2$, $\sigma > 0$, $q_{\alpha,1}$ is the smallest positive root of the equation $j_\alpha(t) = 0$. To formulate one of the main results of this subsection, we need the following weight function, which is a σ-dilation of the weight function introduced in [3, p. 196, (61)]:

$$v(t) = v_\alpha(t) = t^{2\alpha+1} T_{\frac{q_{\alpha,1}}{\sigma}} V(t), \quad t \in \mathbb{R}_+, \tag{3.1}$$

where

$$V(t) = \begin{cases} j_\alpha(\sigma t), & 0 < t < \dfrac{q_{\alpha,1}}{\sigma}; \\ 0, & t \ge \dfrac{q_{\alpha,1}}{\sigma}. \end{cases}$$

Theorem 1. *Let $k \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $\alpha > -1/2$, $\sigma > 0$. For any function $f \in W^r_{2,\mu_\alpha}(B)$ the estimate holds*:

$$E_\sigma(f) \le \frac{\left(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) v(t)\,dt\right)^{k/2}}{\sigma^{2r}\left(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)\,dt\right)^{k/2}},$$

where $q_{\alpha,1}$ is the smallest positive root of the equation $j_\alpha(t) = 0$ and $v(t)$ is the weight function (3.1).

Proof of Theorem 1. For any function $f \in W^r_{2,\mu_\alpha}(B)$ taking into account

$$E_\sigma(f)_{2,\mu_\alpha} = \Big(\int\limits_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\Big)^{1/2}$$

and applying Hölder's inequality with parameter $\frac{1}{p} = \frac{2k-2}{2k}$, $\frac{1}{q} = \frac{1}{k}$ and from elementary transformations, we have

$$E^2_\sigma(f) - \int\limits_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 j_\alpha(\lambda t) d\mu_\alpha(\lambda) = \int\limits_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 (1 - j_\alpha(\lambda t)) d\mu_\alpha(\lambda)$$

$$= \int\limits_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^{2-\frac{2}{k}} |\hat{H}_\alpha(f)(\lambda)|^{\frac{2}{k}} (1 - j_\alpha(\lambda t)) d\mu_\alpha(\lambda)$$

$$\le E_\sigma^{2-\frac{2}{k}}(f) \Big(\int\limits_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 (1 - j_\alpha(\lambda t))^k d\mu_\alpha(\lambda)\Big)^{1/k}$$

$$\le E_\sigma^{2-\frac{2}{k}}(f) \Big(\sigma^{-4r} \int\limits_\sigma^\infty \lambda^{4r} |\hat{H}_\alpha(f)(\lambda)|^2 (1 - j_\alpha(\lambda t))^k d\mu_\alpha(\lambda)\Big)^{1/k}$$

$$\le E_\sigma^{2-\frac{2}{k}}(f) \Big(\sigma^{-4r} \int\limits_0^\infty \lambda^{4r} |\hat{H}_\alpha(f)(\lambda)|^2 (1 - j_\alpha(\lambda t))^k d\mu_\alpha(\lambda)\Big)^{1/k}. \tag{3.2}$$

Since

$$\omega^2_k(B^r f, t) = \sup_{0<h\le t} \int\limits_0^\infty \lambda^{4r} |\hat{H}_\alpha(f)(\lambda)|^2 (1 - j_\alpha(\lambda h))^k d\mu_\alpha(\lambda)$$

then from inequalities (3.2) get

$$E^2_\sigma(f) - \int\limits_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 j_\alpha(\lambda t) d\mu_\alpha(\lambda) \le E_\sigma^{2-\frac{2}{k}}(f) \sigma^{-\frac{4r}{k}} \omega_k^{\frac{2}{k}}(B^r f, t). \tag{3.3}$$

Further multiplying both parts of the inequality (3.3) by the weight function v defined by formula (3.1) and integrating by t from zero to $\frac{2q_{\alpha,1}}{\sigma}$ we obtain

$$\int\limits_0^{\frac{2q_{\alpha,1}}{\sigma}} E^2_\sigma(f) v(t) dt - \int\limits_0^{\frac{2q_{\alpha,1}}{\sigma}} \int\limits_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 j_\alpha(\lambda t) d\mu_\alpha(\lambda) v(t) dt$$

$$\le \sigma^{-\frac{4r}{k}} \int\limits_0^{\frac{2q_{\alpha,1}}{\sigma}} E_\sigma^{2-\frac{2}{k}}(f) \omega_k^{\frac{2}{k}}(B^r f, t) v(t) dt. \tag{3.4}$$

Since the value of the integral (see [3])

$$\int\limits_0^{\frac{2q_{\alpha,1}}{\sigma}} j_\alpha(\lambda t) v(t) dt = \Big(\frac{q_{\alpha,1}}{\sigma}\Big)^{2\alpha+1} \cdot \frac{\sigma}{\lambda^2 - \sigma^2} j'_\alpha(q_{\alpha,1}) j^2_\alpha\Big(\frac{2\lambda q_{\alpha,1}}{\sigma}\Big) \le 0 \tag{3.5}$$

for all $\lambda > \sigma$ then from inequality (3.4) and in view of inequality (3.5) and taking into account the properties of the weight function v and from Lemma A [3] we obtain

$$E_\sigma^2(f)\int\limits_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)\,dt \le \sigma^{-\frac{4r}{k}} E_\sigma^{2-\frac{2}{k}}(f)\int\limits_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f,t)v(t)\,dt.$$

Hence it follows that

$$E_\sigma^{\frac{2}{k}}(f) \le \sigma^{-\frac{4r}{k}} \frac{\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f,t)v(t)\,dt}{\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)\,dt},$$

$$E_\sigma(f) \le \frac{\Big(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f,t)v(t)\,dt\Big)^{k/2}}{\sigma^{2r}\Big(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)\,dt\Big)^{k/2}}. \tag{3.6}$$

Theorem is proved.

Corollary 1. *Let $k = 1,2,\dots, r = 1,2,\dots$, $\alpha > -1/2$ and $\sigma > 0$. For any function $f \in W^r_{2,\mu_\alpha}(B)$ holds inequality*

$$E_\sigma(f) \le \sigma^{-2r}\omega_k\Big(B^r f, \frac{2q_{\alpha,1}}{\sigma}\Big),$$

where $q_{\alpha,1}$ is a smallest positive root of the equation $j_\alpha(t) = 0$ and $v(t)$ is the weight function defined by formula (3.1).

Proof of Corollary 1. Let $f \in W^r_{2,\mu_\alpha}(B)$. Then from inequality (3.6) in virtue by the monotonicity of the modulus of smoothness $\omega_k(B^r f,t)$ it follows

$$E_\sigma(f) \le \sigma^{-2r}\frac{\Big(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f,t)v(t)\,dt\Big)^{k/2}}{\Big(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)\,dt\Big)^{k/2}} \le \sigma^{-2r}\omega_k\Big(B^r f, \frac{2q_{\alpha,1}}{\sigma}\Big).$$

Corollary is proved.

Remark 2. Earlier in [3; 10; 13; 17] were obtained exact constants in the Jackson–Stechkin inequality. In case $r = 0, \alpha = 0, k \in \mathbb{N}$ from the Corollary 1 of Theorem 1 we obtain the result of J. Li, Y.P. Liu [16] with an exact constant in space $L^2([0,1],x)$.

Theorem 2. *Let $\alpha > -1/2$, $q_{\alpha+1,1}$ is the smallest positive zero of the function $j_{\alpha+1}(x)$. Let $k \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $0 < p \le 2$, $\sigma > 0$, $h \in \big(0, \frac{q_{\alpha+1,1}}{\sigma}\big]$. Then the following estimates holds*:

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu\alpha}}{\Big(\displaystyle\int_0^h \omega_k^p(B^r f,t)_{2,\mu_\alpha}dt\Big)^{1/p}} = \frac{1}{\Big(\displaystyle\int_0^h (1-j_\alpha(\sigma t))^{\frac{kp}{2}}\,dt\Big)^{1/p}}.$$

Proof of Theorem 2. Upper estimate. Let $0 < p \le 2$. First of all, we note that (2.3) for an arbitrary function $f \in W^r_{2,\mu_\alpha}(B)$ imply the inequality

$$\omega_k^2(B^r f,t)_{2,\mu_\alpha} \ge \int\limits_\sigma^\infty \lambda^{4r}\,(1-j_\alpha(\lambda t))^k\,|\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda). \tag{3.7}$$

Raising both sides of the inequality (3.7) by the power $p/2$, integrating the variable t over the range $t=0$ to $t=h$, we obtain

$$\Big(\int\limits_0^h \big(\omega_k^2(B^r f,t)_{2,\mu_\alpha}\big)^{p/2}\,dt\Big)^{1/p}$$

$$\geq \Big[\int\limits_0^h \Big(\int\limits_\sigma^\infty \lambda^{4r}(1-j_\alpha(\lambda t))^k |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\Big)^{p/2} dt\Big]^{1/p} = I. \tag{3.8}$$

Further, applying the Minkowski inequality to the right side of inequality (3.8), we obtain for $\dfrac{p}{2}\leq 1$

$$I \geq \Big[\int\limits_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 \Big(\int\limits_0^h \lambda^{2rp}(1-j_\alpha(\lambda t))^{\frac{kp}{2}}dt\Big)^{2/p} d\mu_\alpha(\lambda)\Big]^{1/2}. \tag{3.9}$$

From (3.9) by virtue of the lemma 2 we have

$$I \geq \Big(\int\limits_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\Big)^{1/2} \inf_{\lambda\geq\sigma}\Big\{\lambda^{2rp}\int\limits_0^h (1-j_\alpha(\lambda t))^{\frac{kp}{2}}dt\Big\}^{1/p} =$$

$$= \sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}\Big(\int\limits_0^h (1-j_\alpha(\sigma t))^{\frac{kp}{2}}dt\Big)^{1/p}. \tag{3.10}$$

Thus, from inequalities (3.7) and (3.10) we obtain

$$\Big(\int\limits_0^h \omega_k^p(B^r f,t)_{2,\mu_\alpha}dt\Big)^{1/p} \geq \sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}\Big(\int\limits_0^h (1-j_\alpha(\sigma t))^{\frac{kp}{2}}dt\Big)^{1/p}.$$

Hence it follows that for all $f\in W^r_{2,\mu_\alpha}(B)$ the inequality

$$\frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\Big(\int_0^h \omega_k^p(B^r f,t)_{2,\mu_\alpha}dt\Big)^{1/p}} \leq \frac{1}{\Big(\int_0^h (1-j_\alpha(\sigma t))^{\frac{kp}{2}}dt\Big)^{1/p}}.$$

Now in this inequality we can pass to a supremum over all $f\in W^r_{2,\mu_\alpha}(B)$, then

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\Big(\int_0^h \omega_k^p(B^r f,t)_{2,\mu_\alpha}dt\Big)^{1/p}} \leq \frac{1}{\Big(\int_0^h (1-j_\alpha(\sigma t))^{\frac{kp}{2}}dt\Big)^{1/p}}. \tag{3.11}$$

Thus, the upper estimates are proved.

Let us prove the lower estimates. To obtain a lower estimate, we construct the function $f_\epsilon\in W^r_{2,\mu_\alpha}(B)$ so that:

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\Big(\int_0^h \omega_k^p(B^r f,t)_{2,\mu_\alpha}dt\Big)^{1/p}} \geq \frac{\sigma^{2r}E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\Big(\int_0^h \omega_k^p(B^r f_\epsilon,t)_{2,\mu_\alpha}dt\Big)^{1/p}}.$$

To do this, we use function $f_\epsilon(x)\in W^r_{2,\mu_\alpha}(B)$ constructed by Babenko in [3] and such that

$$\hat{H}_\alpha(f_\epsilon)(\lambda) = \begin{cases} |\lambda|^{-\alpha-\frac{1}{2}} & \text{if} \quad \sigma<|\lambda|<\sigma+\epsilon, \\ 0, & \text{otherwise.}\end{cases} \tag{3.12}$$

Relations (2.1) and (3.12) imply the equalities

$$E_\sigma^2(f_\epsilon)_{2,\mu_\alpha} = \int\limits_\sigma^\infty |\hat{H}_\alpha(f_\epsilon)(\lambda)|^2 d\mu_\alpha(\lambda) = \int\limits_\sigma^{\sigma+\epsilon} |\hat{H}_\alpha(f_\epsilon)(\lambda)|^2 d\mu_\alpha(\lambda) = \epsilon.$$

Therefore

$$E_\sigma(f_\epsilon)_{2,\mu_\alpha} = \sqrt{\epsilon}. \tag{3.13}$$

Using the properties of the Hankel transform [20; 30], we write

$$\hat{H}_\alpha(B^r f_\epsilon)(\lambda) = \lambda^{2r}\hat{H}_\alpha(f_\epsilon)(\lambda)$$

and by virtue of the equality (2.1) and taking into account that (3.12) we have

$$\omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} = \int\limits_\sigma^{\sigma+\epsilon} \lambda^{4r}|\hat{H}_\alpha(f_\epsilon)(\lambda)|^2(1-j_\alpha(\lambda t))^k d\mu_\alpha(\lambda)$$

$$\leq (\sigma+\epsilon)^{4r}(1-j_\alpha((\sigma+\epsilon)t))^k\epsilon. \tag{3.14}$$

Raising the left and right parts of the inequality (3.14) to the power of $\frac{p}{2}$ and integrating both parts of the resulting relation, we have

$$\left(\int\limits_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right)^{1/p} \leq (\sigma+\epsilon)^{2r}\sqrt{\epsilon}\left(\int\limits_0^h (1-j_\alpha((\sigma+\epsilon)t)^{\frac{kp}{2}} dt\right)^{1/p}.$$

Using (3.13), (3.14) we write

$$\frac{\sigma^{2r}E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right)^{1/p}} \geq \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r}\left(\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{\frac{kp}{2}} dt\right)^{1/p}}. \tag{3.15}$$

Since $f_\epsilon \in W_{2,\mu_\alpha}^r(B)$, then from (3.15) and (1.4) we obtain

$$\sup_{f\in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{1/p}} \geq \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r}\left(\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{\frac{kp}{2}} dt\right)^{1/p}}. \tag{3.16}$$

Obviously, the left side of inequality (3.16) does not depend on ϵ, and the expression located on its right side is the function of ϵ (with fixed values of other parameters). Calculating the upper bound for ϵ from the right side of inequality (3.16), we write

$$\sup_{f\in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{1/p}} \geq \frac{1}{\left(\int_0^h (1-j_\alpha(\sigma t))^{\frac{kp}{2}} dt\right)^{1/p}}. \tag{3.17}$$

Comparing the upper estimate (3.11) and the lower estimate (3.17), we obtain the proof of the theorem.

Remark 3. Previously, similar results were obtained in [22]: for $\alpha = 0$ and

in [32] : for $\alpha = -\frac{1}{2}$, $\frac{1}{2r} < p \leq 2$, $j_{-\frac{1}{2}}(\sigma t) = \cos\sigma t$,

in [31] : for $\alpha = \frac{1}{2}$, $0 < p \leq 2$, $j_{\frac{1}{2}}(t) = \frac{\sin t}{t}$, and

in [26] : for $\alpha = \frac{1}{2}$, $\frac{1}{r} < p \leq 2$, $j_{\frac{1}{2}}(t) = \frac{\sin t}{t}$,

in [28] : for $\alpha = -\frac{1}{2}$, $p = 2$ (see 219 p.).

The following theorem is an extension of Ligun's result to the case of the generalized modulus of smoothness $\omega_k(B^r f,t)_{2,\mu_\alpha}$.

Theorem 3. *Let $q_{\alpha+1,1}$ be the smallest positive zero of the function $j_{\alpha+1}(t)$, $\alpha > -\frac{1}{2}$, $k\in\mathbb{N}$, $r\in\mathbb{Z}_+$, $0<p\le 2$, $\sigma>0$, $h\in\big(0,\frac{q_{\alpha+1,1}}{\sigma}\big)$ and $\varphi(t)$ is a function that is measurable, non-negative, integrable on $[0,h]$, and not identically zero. Then the inequalities*

$$\frac{1}{\gamma_{\sigma,k,r,p}(\varphi,h)}\le \Xi_{\sigma,k,r,p}(\varphi,h)\le \frac{1}{\inf\{\gamma_{\lambda,k,r,p}(\varphi,h):\sigma\le\lambda<\infty\}}$$

holds where

$$\gamma_{\lambda,k,r,p}(\varphi,h)=\Big(\lambda^{2rp}\int_0^h(1-j_\alpha(\lambda t))^{\frac{kp}{2}}\varphi(t)dt\Big)^{1/p}.$$

Proof of Theorem 3. Let $0<p\le 2$, then arguing in the same way as in the previous theorem 2 we get

$$\omega_k^2(B^r f,t)_{2,\mu_\alpha}\ge\int_\sigma^\infty \lambda^{4r}(1-j_\alpha(\lambda h))^k|\hat{H}_\alpha(f)(\lambda)|^2d\mu_\alpha(\lambda).$$

Raising both sides of this inequality by the power $p/2$ and multiplying them by $\varphi(t)$, integrating over t from zero to h, we have

$$\Big(\int_0^h\Big(\omega_k^2(B^r f,t)_{2,\mu_\alpha}\Big)^{p/2}\varphi(t)dt\Big)^{1/p}$$

$$\ge\Big[\int_0^h\Big(\int_\sigma^\infty\lambda^{4r}(1-j_\alpha(\lambda t))^k|\hat{H}_\alpha(f)(\lambda)|^2d\mu_\alpha(\lambda)\Big)^{p/2}\varphi(t)dt\Big]^{1/p}=I. \quad (3.18)$$

Applying the Minkowski inequality to the right side of inequality (3.18), we obtain for $\frac{p}{2}\le 1$

$$I\ge\Big[\int_\sigma^\infty|\hat{H}_\alpha(f)(\lambda)|^2d\mu_\alpha(\lambda)\Big(\int_0^h\lambda^{2rp}(1-j_\alpha(\lambda t))^{\frac{kp}{2}}\varphi(t)dt\Big)^{2/p}\Big]^{1/2}$$

$$\ge E_\sigma(f)\inf_{\sigma\le\lambda<\infty}\Big\{\lambda^{2pr}\int_0^h(1-j_\alpha(\lambda t))^{\frac{kp}{2}}\varphi(t)dt\Big\}^{1/p}. \quad (3.19)$$

Thus, combining (3.18) and (3.19) we have

$$\Big(\int_0^h\omega_k^p(B^r f,t)_{2,\mu_\alpha}\varphi(t)dt\Big)^{1/p}\ge E_\sigma(f)\inf_{\sigma\le\lambda<\infty}\{\gamma_{\lambda,k,r,p}(\varphi,h)\}.$$

It follows that for all $f\in W^r_{2,\mu_\alpha}(B)$ the inequality:

$$\frac{E_\sigma(f)}{\Big(\int_0^h\omega_k^p(B^r f,t)_{2,\mu_\alpha}\varphi(t)dt\Big)^{1/p}}\le\frac{1}{\inf\{\gamma_{\lambda,k,r,p}(\varphi,h):\sigma\le\lambda<\infty\}}.$$

Passing to the supremum with respect to all $f\in W^r_{2,\mu_\alpha}(B)$ in this inequality, we obtain

$$\Xi_{\sigma,k,r,p}(\varphi,h)=\sup_{f\in W^r_{2,\mu_\alpha}(B)}\frac{E_\sigma(f)_{2,\mu_\alpha}}{\Big(\int_0^h \omega_k^p(B^rf,t)_{2,\mu_\alpha}\varphi(t)dt\Big)^{1/p}}$$

$$\le\frac{1}{\inf\{\gamma_{\lambda,k,r,p}(\varphi,h)\colon \sigma\le\lambda<\infty\}}.$$

Thus, the upper estimate proved.

To obtain a lower estimate, we construct the function $f_\epsilon\in W^r_{2,\mu_\alpha}(B)$ so that the inequality takes place:

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)}\frac{E_\sigma(f)_{2,\mu_\alpha}}{\Big(\int_0^h \omega_k^p(B^rf,t)_{2,\mu_\alpha}\varphi(t)dt\Big)^{1/p}}\ge\frac{E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\Big(\int_0^h \omega_k^p(B^rf_\epsilon,t)_{2,\mu_\alpha}\varphi(t)dt\Big)^{1/p}}. \tag{3.20}$$

Let us proceed to the proof of the inequality (3.20). Let's take the function $f_\epsilon(x)\in W^r_{2,\mu_\alpha}(B)$ constructed in [3] and such that

$$\hat{H}_\alpha(f_\epsilon)(\lambda)=\begin{cases}|\lambda|^{-\alpha-\frac12} & \text{if}\quad \sigma<|\lambda|<\sigma+\epsilon\\ 0, & \text{otherwise.}\end{cases}$$

Then by using (3.13) and (3.14) we write

$$\frac{E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\Big(\int\limits_0^h \omega_k^p(B^rf_\epsilon,t)_{2,\mu_\alpha}\varphi(t)dt\Big)^{1/p}}\ge\frac{1}{\Big[(\sigma+\epsilon)^{2pr}\int_0^h(1-j_\alpha((\sigma+\epsilon)t))^{\frac{kp}{2}}\varphi(t)dt\Big]^{1/p}}. \tag{3.21}$$

Therefore in virtue of the inequality (3.21) and from the definition (2.4) we obtain

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)}\frac{E_\sigma(f)_{2,\mu_\alpha}}{\Big(\int\limits_0^h \omega_k^p(B^rf,t)_{2,\mu_\alpha}\varphi(t)dt\Big)^{1/p}}\ge\frac{1}{\Big[(\sigma+\epsilon)^{2pr}\int_0^h(1-j_\alpha((\sigma+\epsilon)t))^{\frac{kp}{2}}\varphi(t)dt\Big]^{1/p}}. \tag{3.22}$$

Obviously, the left side of inequality (3.22) does not depend on ϵ, and the expression located on its right side is the function of ϵ (with fixed values of other parameters). Since the left side of inequality (3.22) does not depend on ϵ, then calculating the supremum with respect to ϵ from its right side, we write

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)}\frac{E_\sigma(f)_{2,\mu_\alpha}}{\Big(\int_0^h \omega_k^p(B^rf,t)_{2,\mu_\alpha}\varphi(t)dt\Big)^{1/p}}\ge\frac{1}{\Big(\sigma^{2pr}\int_0^h(1-j_\alpha(\sigma t))^{\frac{kp}{2}}\varphi(t)dt\Big)^{1/p}}. \tag{3.23}$$

Comparing the upper bound (3.19) and the lower bound (3.23), we obtain the required double inequality.

Theorem is proved.

Remark 4. Previously, similar results were obtained in [22] for $\alpha=0$ and
in [32] : for $\alpha=-\dfrac12,\quad \dfrac{1}{2r}<p\le 2,\quad \sigma=n,\quad j_{-\frac12}(nt)=\cos nt,$ and also
in [31; 33] : for $\alpha=\dfrac12,\quad 0<p\le 2,\quad j_{\frac12}(t)=\dfrac{\sin t}{t},$ and
in [32] : for $\alpha=-\dfrac12,\quad 0<p\le 2,\quad \sigma=n,\quad j_{-\frac12}(nt)=\cos nt.$

The corollary follows from the proved the theorem.

Let us find out what differential properties the weight function $\varphi(t)$ must have in order to the following equality holds

$$\inf_{\sigma\le\lambda<\infty}\{\gamma_{\lambda,k,r,p}(\varphi,h)\}=\gamma_{\sigma,k,r,p}(\varphi,h).$$

Corollary 2. *Let $\alpha>-\dfrac{1}{2}$ and the weight function $\varphi(t)$ defined on the segment $[0,h]$ be non-negative and continuously differentiable on it. If for all $t\in[0,h]$ and $p\in(0,2]$, $r\in\mathbb{N}$ inequality*

$$(2rp-1)\varphi(t)-t\varphi'(t)\ge 0 \tag{3.24}$$

holds, then for all $\sigma>0$, $h\in\left(0,\dfrac{q_{\alpha+1,1}}{\sigma}\right)$ we have

$$\inf\{\gamma_{\lambda,k,r,p}(\varphi,h):\sigma\le\lambda<\infty\}=\gamma_{\sigma,k,r,p}(\varphi,h)$$

and

$$\Xi_{\sigma,k,r,p}(\varphi,h)=\frac{1}{\left(\sigma^{2pr}\int_0^h(1-j_\alpha(\sigma t))^{\frac{kp}{2}}\varphi(t)dt\right)^{1/p}}.$$

Proof of Corollary 2. Since $\gamma_{\lambda,k,r,p}(\varphi,h)$ it is sufficient to prove that under the above assumptions on $\varphi(t)$ and the function

$$\zeta(y)=y^{2rp}\int_0^h(1-j_\alpha(yt))^{\frac{kp}{2}}\varphi(t)dt$$

for $y\ge\sigma$ is strictly increasing. Because

$$\zeta'(y)=2rpy^{2rp-1}\int_0^h(1-j_\alpha(yt))^{\frac{kp}{2}}\varphi(t)dt+y^{2rp}\int_0^h\frac{d}{dy}(1-j_\alpha(yt))^{\frac{kp}{2}}\varphi(t)dt, \tag{3.25}$$

then, using the easily verifiable identity

$$\frac{d}{dy}(1-j_\alpha(yt))^{\frac{kp}{2}}=\frac{t}{y}\frac{d}{dt}(1-j_\alpha(yt))^{\frac{kp}{2}} \tag{3.26}$$

from (3.25) and taking into account (3.26) we have

$$\zeta'(y)=2rpy^{2rp-1}\int_0^h(1-j_\alpha(yt))^{\frac{kp}{2}}\varphi(t)dt+y^{2rp-1}\int_0^h\frac{d}{dt}(1-j_\alpha(yt))^{\frac{kp}{2}}(t\varphi(t))dt.$$

Applying the method of integration by parts when calculating the second integral, we come to the conclusion

$$\zeta'(y)=y^{2rp-1}\left[(1-j_\alpha(yh))^{\frac{kp}{2}}h\varphi(h)+\int_0^h(1-j_\alpha(yt))^{\frac{kp}{2}}[(2rp-1)\varphi(t)+t\varphi'(t)]dt\right]. \tag{3.27}$$

Since $|j_\alpha(y)|\le 1$ for all $y\in[0,\infty)$, then by virtue of the (3.24) taking into account the condition $p\in(0,2]$, $r\in\mathbb{N}$ from (3.27) we have $\zeta'(y)\ge 0$ for $y\ge\sigma$. Whence follows $\inf\{\zeta(y):\sigma\le y<\infty\}=\zeta(\sigma)$,which is equivalent to the equality

$$\inf\{\gamma_{\lambda,k,r,p}(\varphi,h):\sigma\le\lambda<\infty\}=\gamma_{\sigma,k,r,p}(\varphi,h).$$

Then, by virtue of the double inequality, from the theorem 3 we obtain the required equality.

Corollary is proved.

Corollary 3. *Let $q_{\alpha+1,1}$ be the smallest positive zero of the function $j_{\alpha+1}(t)$, $k\in\mathbb{N}$, $r\in\mathbb{Z}_+$, $0<p\le 2$, $\sigma>0$, $h\in\big(0,\dfrac{q_{\alpha+1,1}}{\sigma}\big)$, $\alpha>-\dfrac{1}{2}$. Then the estimates*

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)}\frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\Big(\int_0^h\omega_k^p(B^rf,t)_{2,\mu_\alpha}d(1-j_\alpha(\sigma t))\Big)^{1/p}}=\left(\frac{\dfrac{kp}{2}+1}{(1-j_\alpha(\sigma h))^{\frac{kp}{2}+1}}\right)^{1/p}$$

holds.

4. Approximation in $L^2(\mathbb{R}^m)$

The exact Jackson–Stechkin inequality and its various generalizations have become the subject of research for many specialists in the last 50 years.

For the multidimensional case, we only briefly list some classical important sharp results pertaining to the Jackson inequality. The exact Jackson constant in the space $L_2(\mathbb{T}^d)$ ($d\in\mathbb{N}$, $d>1$) was obtained by Yudin [35; 21] (the order of the modulus of continuity is $r=1$) in 1981. Similarly, in the space $L_2(\mathbb{R}^d)$, the exact Jackson inequalities were proved by Popov [21] for the case $d=2,3,\dots$ and $r=1$. In the space $L_2(\mathbb{S}^{d-1})$ ($\mathbb{S}^{d-1}$ is the unit sphere of $\mathbb{R}^d$, $d\ge 3$), the exact constants in the Jackson inequalities were established by Arestov and Popov [2] ($d=3,4,\text{and}\, r\in\mathbb{N}$), and by Babenko [3] ($d\ge 5, r>0$). Berdysheva [4] obtained the exact Jackson inequality in $L_2(\mathbb{R}^d)$ where the definitions of exponential type and modulus of continuity use dilations of the corresponding convex centrally symmetric sets.

Some historical information on Jackson–Stechkin inequalities in $L^2(\mathbb{R}^m)$ are contained in [3; 4; 10; 13; 19; 27].

Let $L^2=L^2(\mathbb{R}^m)$ be the Hilbert space of complex functions on $\mathbb{R}^m$ with inner product and norm

$$(f,g)=\int_{\mathbb{R}^m}f(x)\overline{g(x)}\,dx,\quad \|f\|=\sqrt{(f,f)}.$$

The Fourier transform of the function $f\in L^2$ is defined by the formula

$$\hat f(y)=\frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m}f(x)e^{-ix\cdot y}dx,$$

where $x\cdot y=\sum_{l=1}^m x_l\cdot y_l$ is the scalar product of vectors x and y from $\mathbb{R}^m$. The function f can be expanded in terms of its Fourier transform $\hat f$ as follows:

$$f(x)=\frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m}f(y)e^{ix\cdot y}dy. \tag{4.1}$$

The Fourier transform in the space L^2 satisfies the Plancherelle formula

$$(f,g)=(\hat f,\hat g),\quad f,g\in L^2.$$

Denote by W_σ the class of entire functions of exponential spherical type $\sigma>0$ belonging to the space. The class W_σ of entire functions consists of entire functions $g\in L^2$ supported by $\operatorname{supp}\hat g$ whose Fourier transform lies in the Euclidean ball $B^m_\sigma=\big\{x\in\mathbb{R}^m\colon |x|=\sqrt{(x,x)}\le\sigma\big\}$ of radius $\sigma>0$ and centered at the origin of the space $\mathbb{R}^m$.

The best approximation of a function f from L^2 by the class W_σ is the quantity

$$A_\sigma f=\inf\{\|f-g\|:g\in W_\sigma\}. \tag{4.2}$$

A spherical shift with step h is an operator S_h acting according to the rule (see [3])

$$S_h f(x) = \frac{1}{|\mathbb{S}^{m-1}|} \int\limits_{\mathbb{S}^{m-1}} f(x + h\xi) d\xi,$$

where $\mathbb{S}^{m-1}$ is the unit Euclidean sphere in $\mathbb{R}^m$, $|\mathbb{S}^{m-1}|$ is its surface area.

Let I be the identity operator, k be a positive number. Following H.P. Rustamov [23] the operator $(I - S_h f)^{\frac{k}{2}}$, we will call the difference operator of order k with step h and will be denoted by Δ_h^k:

$$\Delta_h^k = \sum_{l=0}^{\infty} (-1)^l \binom{\frac{k}{2}}{l} S_h^l,$$

and the modulus of continuity of the k-th order of the function $f \in L^2(\mathbb{R}^m)$ is the function of the variable $\tau > 0$:

$$\omega_k(f, \tau) = \sup\big\{\|\Delta_h^k f\| \colon 0 < h \le \tau\big\}.$$

Using the Plancherell formula, it is easy to verify that the value of the best approximation of the function $f \in L^2(\mathbb{R}^m)$ is expressed in terms of

$$A_\sigma^2 f = \int\limits_{|y|>\sigma} |\hat{f}(y)|^2 dy.$$

In this paper, we study the problem of the exact constant $K_\sigma(\tau, k, m)$, $\tau > 0$, $k \ge 1$, $m = 2, 3, \dots$, in the Jackson–Stechkin inequality in the space $L^2(\mathbb{R}^m)$

$$A_\sigma(f) \le K\omega_k(f, \tau), \quad f \in L^2(\mathbb{R}^m).$$

The exact constant in this inequality can be represented as

$$K_\sigma(\tau, k, m) = \sup\Big\{\frac{A_\sigma(f)}{\omega_k(f, \tau)} \colon f \in L^2(\mathbb{R}^m\Big\}.$$

It is known [27, p. 176; 34] that the spherical shift operator S_h with step $h > 0$ acts on the function $e_y(x) = e^{ix\cdot y}$ like this:

$$S_h e_y(x) = \frac{1}{|\mathbb{S}^{m-1}|} \int\limits_{\mathbb{S}^{m-1}} e^{i(x+h\xi)\cdot y} d\xi$$

$$= \frac{e^{ix\cdot y}}{|S^{m-1}|} \int\limits_{S^{m-1}} e^{ih\xi\cdot y} d\xi = j_{\frac{m-2}{2}}(h|y|) e_y(x). \tag{4.3}$$

Applying k times to both sides of equality (4.1) the spherical shift operator and using the relation (4.3) we have

$$S_h^k f(x) = \frac{1}{(2\pi)^{m/2}} \int\limits_{\mathbb{S}^{m-1}} (j_{\frac{m-2}{2}}(h|y|))^k \hat{f}(y) e^{ix\cdot y} dy. \tag{4.4}$$

Based on the definition of the difference operator Δ_h^k, by virtue of (4.4) we obtain

$$\Delta_h^k f(x) = \int\limits_{\mathbb{R}^m} (1 - j_{\frac{m-2}{2}}(h|y|))^{\frac{k}{2}} \hat{f}(y) e^{ix\cdot y} dy. \tag{4.5}$$

Hence, by virtue of the Plancherelle formula, we have from (4.5)

$$\|\Delta_h^k f\|^2 = \int\limits_{\mathbb{R}^m} (1 - j_{\frac{m-2}{2}}(h|y|))^k |\hat{f}(y)|^2 dy.$$

5. Jackson–Stechkin Theorem in $L^2(\mathbb{R}^m)$

In [7] Chernykh noted that since the functional

$$J_k(f,h)=\Big(\frac{n}{2}\int\limits_0^{\frac{\pi}{n}}\omega_k^2(f,t)\sin nt\,dt\Big)^{1/2}$$

less than the Jacksonian functional $\omega_k\big(f,\frac{\pi}{n}\big)$ $(f\neq\text{const})$ and, apparently, it is more natural for characterizing the best approximations $E_{n-1}(f)$ of periodic functions in L_2. For the modulus of smoothness $\omega_k(f,h)_{2,\mu_\alpha}$ we are considering, a similar situation is observed. We will show this in the proof of Corollary 4 of Theorem 4.

Theorem 4. *Let $k\ge 1$, $m\ge 2$, $\sigma>0$. Then for any function $f\in L^2(\mathbb{R}^m)$ we have*

$$A_\sigma(f)\le\frac{\Big(\int_0^{\frac{2q_{\alpha,1}}{\sigma}}\omega_k^{\frac{2}{k}}(B^rf,t)v(t)\,dt\Big)^{k/2}}{\sigma^{2r}\Big(\int_0^{\frac{2q_{\alpha,1}}{\sigma}}v(t)\,dt\Big)^{k/2}},$$

where $q_{\frac{m-2}{2},1}$ is the smallest positive zero of the function $j_{\frac{m-2}{2}}(t)$ and $v(t)=v_{\frac{m-2}{2}}(t)$ is weight function defined by formula (3.1).

Proof of Theorem 4. For any function $f\in L^2(\mathbb{R}^m)$, taking into account the equality

$$A_\sigma(f)=\Big(\int\limits_{|y|>\sigma}|\hat f(y)|^2dy\Big)^{1/2}$$

and by virtue of Hölder's inequality we have

$$A_\sigma^2(f)-\int\limits_\sigma^\infty|\hat f(y)|^2j_{\frac{m-2}{2}}(t|y|)dy=\int\limits_\sigma^\infty|\hat f(y)|^2(1-j_{\frac{m-2}{2}}(t|y|))dy$$

$$=\int\limits_\sigma^\infty|\hat f(y)|^{2-\frac{2}{k}}|\hat f(y)|^{\frac{2}{k}}(1-j_{\frac{m-2}{2}}(t|y|))dy$$

$$\le A_\sigma^{2-\frac{2}{k}}(f)\Big(\sigma^{-4r}\int\limits_\sigma^\infty y^{4r}|\hat f(y)|^2(1-j_{\frac{m-2}{2}}(t|y|))^kdy\Big)^{1/k}. \tag{5.1}$$

Since

$$\omega_k^2(B^rf,t)=\sup_{0\le h\le t}\int\limits_0^\infty y^{4r}|\hat f(y)|^2(1-j_{\frac{m-2}{2}}(h|y|))^kdy$$

then (5.1) implies

$$A_\sigma^2(f)-\int\limits_\sigma^\infty|\hat f(y)|^2j_{\frac{m-2}{2}}(t|y|)dy\le A_\sigma^{2-\frac{2}{k}}(f)\sigma^{-\frac{4r}{k}}\omega_k^{\frac{2}{k}}(B^rf,t). \tag{5.2}$$

Multiplying both sides of the (5.2) inequality by the Babenko weight function $v(t) = v_{\frac{m-2}{2}}(t)$ defined by formula (3.1) and integrating them over t from zero to $\dfrac{2q_{\alpha,1}}{\sigma} = \dfrac{2q_{\frac{m-2}{2},1}}{\sigma}$ we get

$$\int_0^{\frac{2q_{\alpha,1}}{\sigma}} A_\sigma^2(f)v(t)dt - \int_0^{\frac{2q_{\alpha,1}}{\sigma}}\int_\sigma^\infty |\hat{f}(y)|^2 j_{\frac{m-2}{2}}(t|y|)dyv(t)dt$$

$$\leq \sigma^{-\frac{4r}{k}} \int_0^{\frac{2q_{\alpha,1}}{\sigma}} A_\sigma^{2-\frac{2}{k}}(f)\omega_k^{\frac{2}{k}}(B^r f,t)v(t)dt, \tag{5.3}$$

where $q_{\frac{m-2}{2},1}$ is the smallest positive zero of the function $j_{\frac{m-2}{2}}(t)$.

Since in [3] the inequality

$$\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(t|y|)v(t)dt \leq 0 \quad \text{for all} \quad |\mathrm{y}|>1 \tag{5.4}$$

was proved, so from (5.3) by virtue of the inequality (5.4), we obtain

$$A_\sigma^2(f)\int_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)\,dt \leq \sigma^{-\frac{4r}{k}} A_\sigma^{2-\frac{2}{k}}(f)\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f,t)v(t)\,dt.$$

Further, applying the properties of the generalized shift operator $T_h f$ (see [15; 3; 20]) and the function $V(t)$, we have

$$A_\sigma^{\frac{2}{k}}(f) \leq \frac{\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f,t)v(t)\,dt}{\sigma^{\frac{4r}{k}}\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)\,dt}.$$

Hence it follows that

$$A_\sigma(f) \leq \frac{\Big(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f,t)v(t)\,dt\Big)^{k/2}}{\sigma^{2r}\Big(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)\,dt\Big)^{k/2}}.$$

Theorem is proved.

The exact Jackson inequality in the space $L_2(\mathbb{R}^d)$

$$E_\sigma(f)_2 \leq \frac{1}{\sqrt{2}}\omega\big(f, \frac{2q_{d/2-1}}{\sigma}\big)_2 \tag{5.5}$$

It was proved by Ibragimov, Nasibov [12] for $d = 1$ and by Popov [21] for $d = 1, 2, 3$ and by Babenko[3] and by Moskovsky [17] for all d, and by Yudin [36] in $L_2(\mathbb{T}^d)$. The proof of Jackson's inequalities (5.5) in spaces L_2 with an exact constant and an optimal argument in the modulus of continuity is an important line of research on extremal problems in approximation theory. It was proved by Chernykh [7] for $d = 1$, by Moskovsky [18] for $d = 4$, by Gorbachev [9] in general case. In the weighted spaces $L_{2,\alpha}(\mathbb{R}^d)$, the exact Jackson inequalities are known only for $d = 1$. For the case of even functions, they were proved by Babenko [3] and Moskovskii [17].

Corollary 4. *Let* $k \in \mathbb{R}_+$, $k \geq 1$, $\sigma > 0$, $m \geq 2$, $\alpha = \dfrac{m-2}{2}$. *Then for any function* $f \in L^2(\mathbb{R}^m)$ *there is an inequality*

$$A_\sigma(f) \leq \sigma^{-2r}\omega_k\Big(B^r f, \frac{2q_{\alpha,1}}{\sigma}\Big),$$

where $q_{\alpha,1}$ *is the smallest positive zero of the function* $j_\alpha(t)$.

Proof of Corollary 4. Let us first show that the functional

$$G_k\Big(f, \frac{q_{\alpha,1}}{\sigma}\Big) = \frac{\Big(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) v(t)\, dt\Big)^{k/2}}{\sigma^{2r}\Big(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)\, dt\Big)^{k/2}}$$

less than $\omega_k(f, \frac{2q_{\alpha,1}}{\sigma})$. Indeed, the monotonicity of $\omega_k(f,t)$ implies that

$$G_k\Big(f, \frac{2q_{\alpha,1}}{\sigma}\Big) = \frac{\Big(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) v(t)\, dt\Big)^{k/2}}{\sigma^{2r}\Big(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)\, dt\Big)^{k/2}} \leq \sigma^{-2r}\omega_k\Big(B^r f, \frac{2q_{\alpha,1}}{\sigma}\Big). \tag{5.6}$$

From Theorem 4, by virtue of (5.6), we have

$$A_\sigma(f) \leq \frac{\Big(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) v(t)\, dt\Big)^{k/2}}{\sigma^{-2r}\Big(\displaystyle\int_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)\, dt\Big)^{k/2}} = G_k\Big(f, \frac{2q_{\alpha,1}}{\sigma}\Big) \leq \sigma^{-2r}\omega_k\Big(B^r f, \frac{2q_{\alpha,1}}{\sigma}\Big).$$

Corollary 4 is proved.

Remark 5. Earlier in [17; 3; 10; 13] exact constants in the Jackson–Stechkin inequality were obtained. In Corollary 4 of Theorem 4, the exact constant obtained in the inequality Jackson–Stechkin coincides with the exact result of A.G. Babenko [3] for $k \geq 1$. The proof of Corollary 4 of Theorem 4 given here differs from the proof of the theorem of A.G. Babenko [3], A.V. Moskovsky [17], D.V. Gorbachev [9; 10], V. I. Ivanov [13].

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