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ZEROS OF SOLUTIONS OF THIRD-ORDER L–A PAIRS AND LINEARIZABLE ORDINARY DIFFERENTIAL EQUATIONS

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We study the form of the zero lines $x = \varphi(t)$ of simultaneous solutions to an L–A pair of general form composed of an evolution equation $\Psi'_t = \Psi''_{xx}/2 - G(t,x)\Psi$ and an ordinary differential equation $\Psi'''_{xxx} = K(t,x)\Psi''_{xx} + L(t,x)\Psi'_x + M(t,x)\Psi$. It is shown that such lines are given by solutions of a second-order nonlinear ordinary differential equation $\varphi''_{tt} = f(t,\varphi,\varphi'_t)$. Its right-hand side $f(t,\varphi,\varphi'_t)$ is a cubic polynomial in the derivative φ'_t with coefficients explicitly determined from the functions $G(t,x)$, $K(t,x)$, $L(t,x)$, and $M(t,x)$. A procedure for integrating this nonlinear equation is described; in this procedure, initial value problems for two consistent third-order linear ordinary differential equations with independent variables x and t are solved successively, and then the implicit function theorem is applied. It is established that this nonlinear ordinary differential equation belongs to the linearizable class of equations that are reduced by point changes to the equation $\tilde{\varphi}''_{\tilde{t}\tilde{t}} = 0$. These point changes, as was shown in S. Lie's classical work, are explicitly written in terms of simultaneous solutions of two homogeneous systems of third-order linear differential equations with different independent variables. The integration procedures for nonlinear ordinary differential equations described in Lie's work and in the present paper are compared. It is noted that the problem of describing the zeros of simultaneous solutions of similar L–A pairs of higher order is of interest. It is conjectured that the solution of this problem can be connected with an integration procedure for linearizable nonlinear ordinary differential equations of order greater than the second.

Keywords: integrability, simultaneous solutions, ordinary differential equations, nonlinearity, point changes, linearizability.

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