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ON ALMOST UNIVERSAL DOUBLE FOURIER SERIES

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The first examples of universal trigonometric series in the class of measurable functions were constructed by D. E. Men'shov. As follows from Kolmogorov's theorem (the Fourier series of each integrable function in the trigonometric system converges in measure), there is no integrable function whose Fourier series in the trigonometric system is universal in the class of all measurable functions. The author has constructed a function $U \in L^1(\mathbb{T})$, $\mathbb{T} = [-\pi, \pi)$, such that, after an appropriate choice of the signs $\{\delta_k = \pm 1\}_{k=-\infty}^{\infty}$ for its Fourier coefficients, the series $\sum_{k=0}^{\infty} \delta_k \big(a_k(U)\cos kx + b_k(U)\sin kx\big)$ is universal in the class of all measurable functions. The first examples of universal functions were constructed by G. Birkhoff in the framework of complex analysis, where entire functions were represented in any circle by uniformly convergent shifts of the universal function, and by Yu. Martsinkevich in the framework of real analysis, where any measurable function was represented as an almost everywhere limit of some sequence of difference relations of the universal function. In this paper we construct an integrable function $u(x,y)$ of two variables such that, after an appropriate choice of the signs $\{\delta_{k,s} = \pm 1\}_{k,s=-\infty}^{\infty}$ for its Fourier coefficients $\widehat{u}_{k,s}$, the series $\sum_{k,s=-\infty}^{\infty} \delta_{k,s}\widehat{u}_{k,s}e^{i(kx+sy)}$ in the double trigonometric system $\{e^{ikx}, e^{isy}\}_{k,s=-\infty}^{\infty}$ is universal in the class $L^p(\mathbb{T}^2)$, $p \in (0,1)$, and hence in the class of all measurable functions. More precisely, it is established that both rectangular partial sums $S_{n,m}(x,y) = \sum_{|k|\le n}\sum_{|s|\le m} \delta_{k,s}\widehat{u}_{k,s}e^{i(kx+sy)}$ and spherical partial sums $S_R(x,y) = \sum_{k^2+s^2\le R^2} \delta_{k,s}\widehat{u}_{k,s}e^{i(kx+sy)}$ of the series $\sum_{k,s=-\infty}^{\infty} \delta_{k,s}\widehat{u}_{k,s}e^{i(kx+sy)}$ are dense in $L^p(\mathbb{T}^2)$. Recently S. V. Konyagin has proved that there is no function $u \in L^1(\mathbb{T}^d)$, $d \ge 2$, such that the rectangular partial sums of its multiple trigonometric Fourier series are dense in $L^p(\mathbb{T}^2)$, $p \in (0,1)$). This means that the author's result formulated here is, in a sense, final.

Keywords: universal function, universal series, multiple Fourier series in a trigonometric system.

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