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BERNSTEIN–SZEGŐ INEQUALITY FOR TRIGONOMETRIC POLYNOMIALS IN THE SPACE L_0 WITH A CONSTANT GREATER THAN CLASSICAL

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In the set $\mathscr{T}_n$ of trigonometric polynomials f_n of order n with complex coefficients, the Weyl derivative (fractional derivative) $f_n^{(\alpha)}$ of real nonnegative order α is considered. The exact constant $B_n(\alpha,\theta)_p$ in Bernstein–Szegő inequality $\|f_n^{(\alpha)}\cos\theta + \tilde{f}_n^{(\alpha)}\sin\theta\|_p \leqslant B_n(\alpha,\theta)_p\|f_n\|_p$ is analyzed. Such inequalities have been studied for more than 90 years. It is known that, for $1 \leqslant p \leqslant \infty$, $\alpha \geqslant 1$, and $\theta \in \mathbb{R}$, the constant takes the classical value $B_n(\alpha,\theta)_p = n^\alpha$. The case $p = 0$ is of interest at least because the constant $B_n(\alpha,\theta)_0$ takes the maximum value in p for $p \in [0,\infty]$. V. V. Arestov proved that, for $r \in \mathbb{N}$, the Bernstein inequality in L_0 holds with the constant $B_n(r,0)_0 = n^r$, and the constant $B_n(\alpha,\pi/2)_0$ in the Szegő inequality in L_0 behaves as $4^{n+o(n)}$. V. V. Arestov in 1994 and V. V. Arestov and P. Yu. Glazyrina in 2014 studied the question of conditions on the parameters n and α under which the constant in the Bernstein–Szegő inequality takes the classical value n^α. Recently, the author has proved Arestov and Glazyrina's conjecture that the Bernstein–Szegő inequality holds with the constant n^α for $\alpha \geqslant 2n-2$ and all $\theta \in \mathbb{R}$. The question about the exactness of the bound $\alpha = 2n-2$, more precisely, the question of the best constant for $\alpha < 2n-2$ remans open. In the present paper, we prove that for any $0 \leqslant \alpha < n$ one can find $\theta^*(\alpha)$ such that $B_n(\alpha,\theta^*(\alpha))_0 > n^\alpha$.

Keywords: trigonometric polynomials, Weyl derivative, Bernstein–Szegő inequality, space L_0.

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