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ORDER EQUALITIES IN THE SPACES $L_p(\mathbb{T})$, $1<p<\infty$, FOR BEST APPROXIMATIONS AND MODULI OF SMOOTHNESS OF DERIVATIVES OF PERIODIC FUNCTIONS WITH MONOTONE FOURIER COEFFICIENTS

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Denote by $M_p^{(r)}(\mathbb{T})$ the class of all functions $f\in L_p(\mathbb{T})$ whose Fourier coefficients satisfy the conditions: $a_0(f)=0$, $0<n^r a_n(f)\downarrow 0$, and $0<n^r b_n(f)\downarrow 0$ $(n\uparrow\infty)$, where $1<p<\infty$, $r\in\mathbb{N}$, and $\mathbb{T}=(-\pi,\pi]$. We establish order equalities in the class $M_p^{(r)}(\mathbb{T})$ between the best approximations $E_{n-1}(f^{(r)})_p$ by trigonometric polynomials of order $n-1$ and the kth-order moduli of smoothness $\omega_k(f^{(r)};\pi/n)_p$ of rth-order derivatives $f^{(r)}$, on the one hand, and various expressions containing elements of the sequences $\{E_{\nu-1}(f^{(r)})_p\}_{\nu=1}^{\infty}$ and $\{\omega_l(f;\pi/\nu)_p\}_{\nu=1}^{\infty}$, where $l,k\in\mathbb{N}$ and $l>r$, on the other hand. The main results obtained in the present paper can be briefly described as follows. A necessary and sufficient condition for a function f from $M_p^{(r)}(\mathbb{T})$ to lie in the class $L_p^{(r)}(\mathbb{T})$ (this class consists of all functions $f\in L_p(\mathbb{T})$ with absolutely continuous $(r-1)$th derivatives $f^{(r-1)}$ and $f^{(r)}\in L_p(\mathbb{T})$; here $f^{(0)}\equiv f$ and $L_p^{(0)}(\mathbb{T})\equiv L_p(\mathbb{T})$) is that one of the following equivalent conditions is satisfied: $E(f;p;r):=\big(\sum_{n=1}^{\infty} n^{pr-1}E_{n-1}^p(f)_p\big)^{1/p}<\infty \Leftrightarrow \Omega(f;p;l;r):=\big(\sum_{n=1}^{\infty} n^{pr-1}\omega_l^p(f;\pi/n)_p\big)^{1/p}<\infty \Leftrightarrow \sigma(f;p;r):=\big(\sum_{n=1}^{\infty} n^{pr+p-2}(a_n(f)+b_n(f))^p\big)^{1/p}<\infty$. Moreover, the following order equalities hold:

(a) $E(f;p;r)\asymp\|f^{(r)}\|_p\asymp\sigma(f;p;r)\asymp\Omega(f;p;l;r)$;

(b) $E_{n-1}(f^{(r)})_p\asymp n^r E_{n-1}(f)_p+\big(\sum_{\nu=n+1}^{\infty}\nu^{pr-1}E_{\nu-1}^p(f)_p\big)^{1/p},\ n\in\mathbb{N}$;

(c) $\omega_k(f^{(r)};\pi/n)_p\asymp n^{-k}\big(\sum_{\nu=1}^{n}\nu^{p(k+r)-1}E_{\nu-1}^p(f)_p\big)^{1/p}+\big(\sum_{\nu=n+1}^{\infty}\nu^{pr-1}E_{\nu-1}^p(f)_p\big)^{1/p},\ n\in\mathbb{N}$;

(d) $E_{n-1}(f^{(r)})_p+n^r\omega_l(f;\pi/n)_p\asymp\big(\sum_{\nu=n+1}^{\infty}\nu^{pr-1}\omega_l^p(f;\pi/\nu)_p\big)^{1/p}\asymp$
$\asymp\omega_k(f^{(r)};\pi/n)_p+n^r\omega_l(f;\pi/n)_p,\ n\in\mathbb{N},\ l<k+r$;

(e) $n^{-(l-r)}\big(\sum_{\nu=1}^{n}\nu^{p(l-r)-1}E_{\nu-1}^p(f^{(r)})_p\big)^{1/p}\asymp\big(\sum_{\nu=n+1}^{\infty}\nu^{pr-1}\omega_l^p(f;\pi/\nu)_p\big)^{1/p}\asymp$
$\asymp n^{-(l-r)}\big(\sum_{\nu=1}^{n}\nu^{p(l-r)-1}\omega_k^p(f^{(r)};\pi/\nu)_p\big)^{1/p},\ n\in\mathbb{N},\ l<k+r$;

(f) $\omega_k(f^{(r)};\pi/n)_p\asymp\big(\sum_{\nu=n+1}^{\infty}\nu^{pr-1}\omega_l^p(f;\pi/\nu)_p\big)^{1/p},\ n\in\mathbb{N},\ l=k+r$;

(g) $\omega_k(f^{(r)};\pi/n)_p\asymp n^{-k}\big(\sum_{\nu=1}^{n}\nu^{p(k+r)-1}\omega_l^p(f;\pi/\nu)_p\big)^{1/p}+\big(\sum_{\nu=n+1}^{\infty}\nu^{pr-1}\omega_l^p(f;\pi/\nu)_p\big)^{1/p}$,
$n\in\mathbb{N}$, $l>k+r$.

In the general case, one cannot drop the term $n^r\omega_l(f;\pi/n)_p$ in item (d) either in the lower estimate on the left-hand side (for $l>r$) or in the upper estimate on the right-hand side (for $r<l<k+r$). However, if $\{E_{n-1}(f)_p\}_{n=1}^{\infty}\in B_l^{(p)}$ $(\Rightarrow\{E_{n-1}(f^{(r)})_p\}_{n=1}^{\infty}\in B_{l-r}^{(p)})$ or $\{\omega_l(f;\pi/n)_p\}_{n=1}^{\infty}\in B_l^{(p)}$ $(\Rightarrow\{\omega_k(f^{(r)};\pi/n)_p\}_{n=1}^{\infty}\in B_{l-r}^{(p)})$, where $B_l^{(p)}$ is the class of all sequences $\{\varphi_n\}_{n=1}^{\infty}$ $(0<\varphi_n\downarrow 0$ as $n\uparrow\infty)$ satisfying the Bari $(B_l^{(p)})$-condition: $n^{-l}\big(\sum_{\nu=1}^{n}\nu^{pl-1}\varphi_\nu^p\big)^{1/p}=\mathcal{O}(\varphi_n)$, $n\in\mathbb{N}$, which is equivalent to the Stechkin (S_l)-condition, then

$$E_{n-1}(f^{(r)})_p\asymp\bigg(\sum_{\nu=n+1}^{\infty}\nu^{pr-1}\omega_l^p\Big(f;\frac{\pi}{\nu}\Big)_p\bigg)^{1/p}\asymp\omega_k\Big(f^{(r)};\frac{\pi}{n}\Big)_p,\quad n\in\mathbb{N}.$$

Keywords: best approximation, modulus of smoothness, direct and inverse theorems with derivatives of the theory of approximation of periodic functions, trigonometric Fourier series with monotone coefficients, order equalities.

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