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PURSUIT–EVASION PROBLEMS UNDER NONLINEAR INCREASE OF THE PURSUER'S RESOURCE

B. T. Samatov, B. I. Juraev

In the paper, we investigate pursuit-evasion problems in a simple motion differential game with two players, termed a pursuer and an evader. We put different kinds of non-stationary integral constraints, which restrict the energy consumption rate of the players. On the other hand, it is assumed that at each time the players have some additional amount of control resource. The integral constraint on the control of the pursuer is given under certain conditions, which include a non-stationary integral constraint. Firstly, the reachable set of each player is determined. We put forward the parallel approach strategy, which is known as a Π-strategy, for the pursuer, and as a result, we get necessary and sufficient conditions of capture. To solve the evasion problem, a specific admissible strategy is provided for the evader and a sufficient condition is obtained. Furthermore, in the pursuit problem, an optimal capture time is found through the strategy of the evader. In order to illustrate the obtained results, several examples are given, where guaranteed capture times are proposed for the pursuit problems and lower bounds for the distances between the players are obtained for the evasion problem. This work extends the results and methods from the works of R.Isaacs, L.A.Petrosjan, N.N.Krasovskii, A.A.Chikrii, A.A.Azamov, and other authors.

Keywords: pursuit-evasion differential games, simple motion, non-stationary integral constraint, pursuer, evader, strategy, guaranteed capture time.

Б. Т. Саматов, Б. И. Жураев. Задачи преследования-уклонения при нелинейном увеличении ресурса преследователя.

Исследуются задачи преследования-уклонения в дифференциальной игре с простым движением и двумя игроками, называемыми преследователем и убегающим. На управления игроков накладываются различные типы нестационарных интегральных ограничений, связанных со скоростью расходования энергии. Интегральное ограничение на управление преследователя задано при определенных условиях и включает в себя нестационарное интегральное ограничение. Управление убегающего подчиняется геометрическому ограничению. Во-первых, найдено множество достижимости каждого игрока. При использовании преследователем стратегии параллельной сходимости, известной как П-стратегия, получены необходимые и достаточные условия поимки. Для решения задачи уклонения указана конкретная допустимая стратегия убегающего и получено достаточное условие уклонения. Далее, с помощью стратегии убегающего в задаче преследования найдено оптимальное время поимки. Для иллюстрации полученных результатов приводится несколько примеров с численными решениями и рисунками. Настоящая работа дополняет результаты и методы работ Р. Айзекса, Л. А. Петросяна, Н. Н. Красовского, А. А. Чикрия, А. А. Азамова и других ученых, включая авторов данной статьи.

Ключевые слова: дифференциальные игры, преследователь, убегающий, интегральное ограничение, стратегия, преследование, уклонение, гарантированное время поимки, оптимальное время поимки.

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Introduction

The pursuit–evasion problems began to be studied systematically by the American mathematician Rufus Isaacs in the 1950s. His studies were published in the form of monograph [16], which contained many interesting examples of differential games. The author regarded them as problems of variational calculus and tried to apply the Hamilton–Jacobi method, now known as Isaacs' method. But the subject turned out to be far more complicated for the classical methods. The idea used by Isaacs had a heuristic character only. In the 1960s, fundamental results of the theory of differential games were obtained by Pontryagin [24], Krasovskii [18], Bercovitz [5], Subbotin and Chentsov [33], Fleming [9], Friedman [10], and others.

A great amount of the works on differential games that have ever been published consider the cases where the control functions are subject to integral constraints. Integral constraints represent restrictions on resources, energy, power, fuel, and so on. The papers [4; 21; 22] considered linear differential games with integral constraints from the standpoint of Pontryagin's first direct method [24]. A more thorough approach, based on Krasovskii's extreme aiming method for solving differential games with integral constraints, was developed by Krasovskii et al. [19] and then in [8; 17; 34] and other papers. In [7; 31], linear differential pursuit games with integral constraints were investigated by the method of resolving functions and sufficient conditions for the completion of the pursuit were obtained. The positional method of approach for the regular case was transferred to the case of integral constraints in the work of Pshenichnii and Onopchuk [25] and was continued in the works [6; 8] and others for games with different types of constraints. The works [14; 15; 27; 32] are devoted to the study the pursuit and evasion differential games with many pursuers and one evader under integral constraints.

In [11], a control problem with disturbances is examined for a linear dynamical system with delay in the control. The works [12; 13] were devoted to the construction of reachable sets for linear and nonlinear control systems when the controls are subject to quadratic integral constraints. In the works [20; 35], a linear differential game of pursuit when integral constraints on the control functions was studied for the case of the presence of delay, and sufficient conditions of capture were defined.

The desire for greater adequacy of mathematical models with practical problems has led to the need to study differential games with different constraints on the players' controls. In the works [2; 3; 8; 26; 28; 29], differential pursuit-evasion games where different constraints are imposed on the players' controls, were studied.

In the work [30] of Samatov et al., a differential game with non-stationary integral constraints was first examined and the Π-strategy of the pursuer, which pursuit can be completed from a given initial point, was constructed. A sufficient condition of capture was determined. To solve an evasion problem, a particular strategy for the evader was proposed, and using this strategy, necessary and sufficient conditions of evasion were found.

In this paper, a pursuit-evasion differential game with two players is studied. It is assumed that the control functions of the evaders and the pursuers are subject to non-stationary integral constraints; i.e., it is assumed that at each time the players have some additional amount of control resource. Both the evader and the pursuer use simple motions. The capture is considered possible if the pursuer captures the evader at a finite time, and the evasion is considered possible if the evader is not captured by the pursuer. To solve the pursuit problem, the parallel approach strategy (the Π-strategy [1; 23]) will be constructed for the pursuer. To solve the evasion problem, the evader is offered a special admissible strategy and sufficient conditions for the evasion are found.

1. Statement of the problems

Suppose that in $\mathbb{R}^n$ a controlled object P called Pursuer chases another object E called Evader. Denote by x the position of Pursuer and by y the position of Evader in $\mathbb{R}^n$. Let the objects move in accordance with the equations

$$P: \quad \dot{x} = u, \quad x(0) = x_0, \tag{1.1}$$

$$E: \quad \dot{y} = v, \quad y(0) = y_0, \tag{1.2}$$

where $x, y, x_0, y_0, u, v \in \mathbb{R}^n$, $t \in \mathbb{R}_+ := [0, +\infty)$, $n \geq 2$; x_0, y_0 are the initial positions of the objects P and E, respectively, $x_0 \neq y_0$; u and v are the velocity vectors which serve as parameters of the equations.

In equation (1.1), the temporal variation of u must be a measurable function $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$, and this vector function is subject to the constraint

$$\int_0^t |u(s)|^2 ds \le \alpha \int_0^t \varphi(s)ds \quad \text{for } t \ge 0, \tag{1.3}$$

where α is a positive number and $\varphi(t)$ satisfies the conditions

$$\left.\begin{array}{ll} \text{(a)} & \varphi(\cdot) \text{ is continuous and strongly decreasing on the interval } [0,\infty); \\ \text{(b)} & \varphi(0)=1, \quad \varphi(t)>0 \ \text{ for } \ t>0, \ \text{ and } \ \varphi(t)\to 0 \ \text{ as } \ t\to+\infty; \\ \text{(c)} & \int_0^t \varphi(s)ds \le t. \end{array}\right\} \tag{1.4}$$

Further, we denote by U_I the class of all measurable functions $u(\cdot)$ for which the constraint (1.3), (1.4) holds.

Similarly, in equation (1.2), the velocity vector v of Evader as a function of time is a measurable function $v(\cdot): \mathbb{R}_+ \to \mathbb{R}^n$ satisfying the constraint

$$\int_0^t |v(s)|^2 ds \le \sigma t, \quad t \ge 0, \tag{1.5}$$

where σ is a positive number. Further, we denote by V_I the class of all measurable functions $v(\cdot)$ for which the constraint (1.5) is valid.

Definition 1. Functions $u(\cdot) = (u_1(\cdot), u_2(\cdot), \ldots, u_n(\cdot)) \in U_I$ and $v(\cdot) = (v_1(\cdot), v_2(\cdot), ..., v_n(\cdot)) \in V_I$ are called admissible controls of the players P and E, respectively.

Depending on the equations (1.1), (1.2), the pairs $(x_0, u(\cdot))$ and $(y_0, v(\cdot))$, where $u(\cdot) \in U_I$ and $v(\cdot) \in V_I$, generate the trajectories

$$x(t) = x_0 + \int_0^t u(s)ds, \tag{1.6}$$

$$y(t) = y_0 + \int_0^t v(s)ds \tag{1.7}$$

of the players P and E, respectively.

In the differential game (1.1)–(1.5), the goal of Pursuer P is to catch Evader E (the pursuit problem) at some time δ, $0 < \delta < +\infty$, i.e., provide the equality $x(\delta) = y(\delta)$ for (1.6) and (1.7), where $x(t)$ and $y(t)$ are the trajectories generated during the game. The notion of "trajectories generated during the game" needs to be clarified. Evader E tries to avoid meeting Pursuer P (the evasion problem), i.e., to guarantee the relation $x(t) \ne y(t)$ for (1.6) and (1.7) on the time interval $[0, +\infty)$, and, if it is impossible, to postpone the moment of the meeting as far as possible. Naturally, this is a preliminary setting of the problems.

Now let us introduce the notation $z(t) = x(t) - y(t)$, $z_0 = x_0 - y_0$.

Definition 2. We call a function $\boldsymbol{u}: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ a strategy of Pursuer if

(a) $\boldsymbol{u}(t, v)$ is a Lebesgue measurable function with respect to t for each fixed v and a Borel measurable function with respect to v for each fixed t;

(b) $u(\cdot) = \boldsymbol{u}(\cdot, v(\cdot)) \in U_I$ is satisfied for any $v(\cdot) \in V_I$;

(c) if $v_1(\cdot), v_2(\cdot) \in V_I$ and satisfy the equality $v_1(\tau) = v_2(\tau)$ a.e. (almost everywhere) on $[0, t]$, then $u_1(\tau) = u_2(\tau)$ a.e. on $[0, t]$, where $u_i(\cdot) = \boldsymbol{u}(\cdot, v_i(\cdot))$, $i = 1, 2$.

Definition 3. We call a strategy $\boldsymbol{u}(t, v)$ a parallel pursuit strategy or, briefly, a Π-strategy if, for an arbitrary control $v(\cdot) \in V_I$ of Evader, the solution $z(t)$ of the initial value problem

$$\dot{z} = \boldsymbol{u}(t, v(t)) - v(t), \quad z(0) = z_0$$

can be written as

$$z(t) = z_0 C(t, v(\cdot)), \quad C(0, v(\cdot)) = 1,$$

where $C(t, v(\cdot))$ is a scalar function of t, $t \geq 0$. We usually term it as the convergence function in the pursuit problem.

Definition 4. We say that a strategy $\boldsymbol{u}(t, v)$ guarantees capture by a time $T(\boldsymbol{u})$ if, for any control $v(t)$, $t \geq 0$, we have $x(\tau) = y(\tau)$ at some time $\tau \in [0, T(\boldsymbol{u})]$, where $(x(\cdot), y(\cdot))$ is the solution of the initial value problem

$$\dot{x} = \boldsymbol{u}(t, v(t)), \quad x(0) = x_0, \tag{1.8}$$

$$\dot{y} = v(t), \quad y(0) = y_0, \tag{1.9}$$

where $t \in [0, +\infty)$.

Definition 5. An admissible control function $\boldsymbol{v}(t) : \mathbb{R}_+ \to \mathbb{R}^n$ is defined as a strategy for Evader E if, for any control $u(\cdot) \in U_I$, the relation $x(t) \neq y(t)$ is retained for each $t \in [0, +\infty)$, where $x(t)$ and $y(t)$ are solutions of the initial value problems

$$\dot{x} = u(t), \quad x(0) = x_0, \tag{1.10}$$

$$\dot{y} = \boldsymbol{v}(t), \quad y(0) = y_0. \tag{1.11}$$

Lemma 1. *For any control $u(\cdot) \in U_I$, the corresponding solution of equation* (1.1) *satisfies the inclusion $x(t) \in S_{\mu(t)}$ for any $t \in [0, +\infty)$, where $\mu(t) = \sqrt{\alpha} t$ and $S_{\mu(t)}(x_0)$ is the closed ball of the space $\mathbb{R}^n$ with radius $\mu(t)$ centered at the point x_0.*

Proof. Let $u(\cdot) \in U_I$. Applying Cauchy–Bunyakovskii's inequality and taking into consideration inequality (1.3) and condition (c) in (1.4), we have from (1.6) the estimate

$$|x(t) - x_0| \leq \int_0^t |u(s)| ds \leq \sqrt{t} \sqrt{\int_0^t |u(s)|^2 ds} \leq \sqrt{\alpha} t$$

for all t, $t \geq 0$.

Lemma 2. *For any control $v(\cdot) \in V_I$, the corresponding solution of equation* (1.2) *satisfies the inclusion $y(t) \in S_{\lambda(t)}$ for any $t \in [0, +\infty)$, where $\lambda(t) = \sqrt{\sigma} t$.*

Proof. The proof is similar to the proof of Lemma 1. □

In the differential game (1.1)–(1.5), we will study the following problems:

Problem 1. The pursuit problem (in short, the I-game of pursuit).

Problem 2. The evasion problem (in short, the I-game of evasion).

2. The main results

2.1. Definition of the Π_I-strategy

Proposition 1. *If $\alpha > \sigma$, then the equation $\varphi(t) = \dfrac{\sigma}{\alpha}$ has a unique solution T_d on the time interval* $(0, +\infty)$.

Proof. The proof follows from (a) and (b) in (1.4). □

Definition 6. The function

$$\boldsymbol{u}(t,v) = v - c(t,v)\xi_0 \tag{2.1}$$

is called the Π_I-strategy of Pursuer in the I-game of pursuit (1.1)–(1.5), where

$$c(t,v) = \langle v, \xi_0\rangle + \sqrt{\langle v, \xi_0\rangle^2 + \alpha\varphi(t) - \sigma}, \tag{2.2}$$

$\xi_0 = z_0/|z_0|$, and $\langle v, \xi_0\rangle$ denotes the inner product of the vectors v and ξ_0 in $\mathbb{R}^n$.

Proposition 2. *If $\alpha > \sigma$, then for each pair $(t, v) \in [0, T_d] \times \mathbb{R}^n$*
(a) *the scalar function* (2.2) *is well defined, continuous and nonnegative*;
(b) *for $\boldsymbol{u}(t,v)$ the equality*

$$|\boldsymbol{u}(t,v)|^2 = |v|^2 + \alpha\varphi(t) - \sigma \tag{2.3}$$

holds.

Proof. (a) It follows from the definition of the scalar function $c(t, v)$ (see (2.2)) and from Proposition 1 that condition (a) is satisfied.

(b) From (2.1) and (2.2) we obtain

$$|\boldsymbol{u}(t,v)|^2 = \big\langle \boldsymbol{u}(t,v), \boldsymbol{u}(t,v)\big\rangle = \langle v - c(t,v)\xi_0, v - c(t,v)\xi_0\rangle = |v|^2 + c(t,v)\big[c(t,v) - 2\langle v, \xi_0\rangle\big]$$
$$= |v|^2 + \alpha\varphi(t) - \sigma.$$

This completes the proof of Proposition 2.

2.2. Solution of the pursuit problem

Consider the function

$$\Gamma(t) = t\left(\sqrt{\alpha\varphi(t)} - \sqrt{\sigma}\right), \quad t \in [0, T_d]. \tag{2.4}$$

It is obvious that the function $\Gamma(t)$ is continuous on the interval $[0, T_d]$, $\Gamma(0) = 0$, and $\Gamma(T_d) = 0$. Hence it reaches its maximum on the closed and bounded interval $[0, T_d]$. Let

$$\Gamma_* = \Gamma(t_*) = \max_{t\in[0,T_d]} \Gamma(t) \tag{2.5}$$

and $t_* \in [0, T_d]$.

Proposition 3. *If $\alpha > \sigma$, then $\Gamma(t) > 0$ for all $t \in (0, T_d)$.*

Proof. From the definition of the function $\varphi(t)$ (see (1.4)), we have $\dfrac{\sigma}{\alpha} < \varphi(t) < 1$ for all $t \in (0, T_d)$, which proves Proposition 3. □

Theorem 1. *In the I-game of pursuit* (1.1)–(1.5)*, if $\alpha > \sigma$ and $\Gamma_* \geq |z_0|$, then the Π_I-strategy of Pursuer* (2.1) *guarantees capture on the time interval* $[0, T_g]$*, where*

$$T_g = \min\{t \in [0, T_d] : \Gamma(t) = |z_0|\}. \tag{2.6}$$

Proof. Let Evader choose an arbitrary control $v(\cdot) \in V_I$ and let Pursuer realize the Π_I-strategy (2.1). According to equations (1.8) and (1.9), the pairs $(x_0, \boldsymbol{u}(t, v(t))$ and $(y_0, v(t))$ in the I-game of pursuit generate the trajectories

$$x(t) = x_0 + \int_0^t \boldsymbol{u}(s, v(s))ds, \quad y(t) = y_0 + \int_0^t v(s)ds \tag{2.7}$$

of Pursuer and Evader, respectively. Let $z(t) = x(t) - y(t)$, $z(0) = z_0$. Then from (2.7) we have

$$z(t) = z_0 + \int_0^t [\boldsymbol{u}(s, v(s)) - v(s)]ds.$$

Hence from (2.1) we get

$$z(t) = z_0 C(t, v(\cdot)), \tag{2.8}$$

where

$$C(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \int_0^t c(s, v(s))ds. \tag{2.9}$$

Let us estimate the function $C(t, v(\cdot))$ from above for each $t \in [0, T_g]$ and for all $v(\cdot) \in V_I$. For this, from the form (2.2) and from the definition of the function $\varphi(t)$ we have

$$C(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \int_0^t \left[\sqrt{\langle v(s), \xi_0 \rangle^2 + \alpha\varphi(s) - \sigma} + \langle v(s), \xi_0 \rangle \right] ds$$

$$\leq 1 - \frac{1}{|z_0|} \int_0^t \left[\sqrt{\ |v(s)|^2 + \alpha\varphi(s) - \sigma} - |v(s)| \right] ds$$

$$\leq 1 - \frac{1}{|z_0|} \int_0^t \left[\sqrt{\ |v(s)|^2 + \alpha\varphi(t) - \sigma} - |v(s)| \right] ds$$

or

$$C(t, v(\cdot)) \leq 1 - \frac{1}{|z_0|} \int_0^t \left[\sqrt{\ |v(s)|^2 + \alpha\varphi(t) - \sigma} - |v(s)| \right] ds. \tag{2.10}$$

In (2.10), we take $w = |v(s)|,\ 0 \leq s \leq t,\ \zeta(t) = \alpha\varphi(t) - \sigma$. Then we have the function $f(w) = \sqrt{w^2 + \zeta(t)} - w$, and here $\dfrac{d^2 f(w)}{dw^2} > 0$; i.e., $f(w)$ is a convex function in $w \geq 0$. Then from Jensen's inequality

$$tf\left(\frac{1}{t}\int_0^t w(s)ds\right) \leq \int_0^t f(w(s))ds$$

for (2.10) we get the inequality

$$C(t, v(\cdot)) \leq 1 - \frac{t}{|z_0|}\left[\sqrt{\left(\frac{1}{t}\int_0^t |v(s)|ds\right)^2 + \zeta(t)} - \frac{1}{t}\int_0^t |v(s)|ds\right]. \tag{2.11}$$

If we take

$$\varsigma = \varsigma(v_t(\cdot)) = \frac{1}{t}\int_0^t |v(s)|ds,$$

where $v_t(\cdot) = \{v(s)\colon 0 \le s \le t\} \in V_I$ is considered as a variable, then from the expression in square brackets on the right-hand side of (2.11), we have the function $f(\varsigma) = \sqrt{\varsigma^2 + \zeta(t)} - \varsigma$ which is monotonically decreasing with respect to ς. Then from the Cauchy–Bunyakovskii inequality

$$\int_0^t |v(s)|ds \le \sqrt{t}\Big(\int_0^t |v(s)|^2 ds\Big)^{1/2}$$

and from (2.11) we get the estimate

$$C(t, v(\cdot)) \le 1 - \frac{t}{|z_0|}\left[\sqrt{\frac{1}{t}\int_0^t |v(s)|^2 ds + \zeta(t)} - \sqrt{\frac{1}{t}\int_0^t |v(s)|^2 ds}\,\right].$$

Therefore, from (1.5) we obtain

$$C(t, v(\cdot)) \le C(t), \tag{2.12}$$

where $C(t) = 1 - \dfrac{\Gamma(t)}{|z_0|}$. According to (2.6) we have $\Gamma(T_g) = |z_0|$, and hence $C(T_g) = 0$. Then (2.12) implies that

$$C(T_g, v(\cdot)) \le 0 \tag{2.13}$$

for every $v(\cdot) \in V_I$. From (2.9) we find that $C(0, v(\cdot)) = 1$ and the function $C(t, v(\cdot))$ is continuous and decreasing for every $v(\cdot) \in V_I$ on $[0, T_g]$. As a result, from (2.13) it follows that there exists some time $t^* \in [0, T_g]$ for which $C(t^*, v(\cdot)) = 0$ and therefore, by virtue of (2.8), we have $z(t^*) = 0$ or $x(t^*) = y(t^*)$.

Now, prove the admissibility of the strategy (2.1) for all $t \in [0, t^*]$. Let a control $v(\cdot) \in V_I$ be arbitrarily chosen on $[0, t^*]$. Then, by virtue of (1.5) and (2.3), we obtain the inequality

$$\int_0^t |\boldsymbol{u}(s, v(s))|^2 ds = \int_0^t |v(s)|^2 ds + \alpha\int_0^t \varphi(s)ds - \sigma t \le \alpha\int_0^t \varphi(s)ds,$$

which implies that inequality (1.3) is satisfied for every $v(\cdot) \in V_I$ and $t \in [0, t^*]$. This completes the proof of Theorem 1. □

2.3. Solution of the evasion problem

First, we will define a strategy for Evader that solves the evasion problem.

Definition 7. In the I-game of evasion (1.1)–(1.5), the function

$$\boldsymbol{v}(t) = -\sqrt{\sigma}\xi_0, \quad \xi_0 = \frac{z_0}{|z_0|} \tag{2.14}$$

is called the strategy of Evader.

Theorem 2. *In the I-game (1.1)–(1.5),*

(a) *if $\alpha > \sigma$, then the strategy (2.14) guarantees evasion in the time interval $[0, T_o)$, where $T_o = \dfrac{|z_0|}{\sqrt{\alpha} - \sqrt{\sigma}}$;*

(b) *if $\alpha \le \sigma$, then the strategy (2.14) guarantees evasion in the time interval $[0, +\infty)$.*

Proof.

(a) Let Pursuer apply an arbitrary control $u(\cdot) \in U_I$ and let Evader use the strategy (2.14) in the time interval $[0, T_o)$. According to equations (1.10), (1.11), and (2.14), the pairs $(x_0, u(t))$ and $(y_0, \boldsymbol{v}(t))$ in the I-game (1.1)–(1.5) generate the trajectories

$$x(t) = x_0 + \int_0^t u(s)ds, \quad y(t) = y_0 - \int_0^t \sqrt{\sigma}\xi_0 ds$$

for each $t \in [0, T_o)$. Hence

$$z(t) = z_0 + \int_0^t u(s)ds + \int_0^t \sqrt{\sigma}\xi_0 ds,$$

where $z(t) = x(t) - y(t)$, $z_0 = x_0 - y_0$. Let us estimate from below the distance between the players:

$$|z(t)| = \left| z_0 + \int_0^t \sqrt{\sigma}\xi_0 ds + \int_0^t u(s)ds \right| \geq |z_0| + \sqrt{\sigma}t - \int_0^t |u(s)|ds. \tag{2.15}$$

From the Cauchy–Bunyakovskii inequality and from the constraints (1.3), (1.4) we obtain

$$\int_0^t |u(s)|ds \leq \sqrt{t}\Big(\int_0^t |u(s)|^2 ds\Big)^{1/2} \leq \sqrt{\alpha}t.$$

Consequently, from (2.15) we have

$$|z(t)| \geq \Upsilon(t), \tag{2.16}$$

where $\Upsilon(t) = |z_0| + t(\sqrt{\sigma} - \sqrt{\alpha})$. Since $\Upsilon(t) > 0$ for all $t \in [0, T_o)$, from (2.16) it follows that $|z(t)| > 0$, i.e., $x(t) \neq y(t)$ in the interval $[0, T_o)$.

(b) Let $\alpha \leq \sigma$. Then we come to the estimate (2.16) again. We can see that $\Upsilon(t) \geq |z_0|$ for all $t \geq 0$ and, thereby, from (2.16) it follows that $|z(t)| > 0$ or $x(t) \neq y(t)$. This completes the proof of Theorem 2.

Remark. If $\alpha > \sigma$, then $T_o \leq T_g$, which follows from the definition of the function $\varphi(t)$ in (1.4).

3. Examples

3.1. Example 1

Let in the I-game (1.1)–(1.5), $\varphi(t) = \dfrac{1}{t+1}$ and $\alpha = 9$, $\sigma = 1$, $|z_0| = 1$. Here it is easy to check the fulfillment of all the conditions in (1.4) for the function $\varphi(t)$. Then from (2.4) it follows that $\Gamma(t) = t\Big(\dfrac{3}{\sqrt{t+1}} - 1\Big)$. According to Proposition 1 and (2.5), we get $T_d = 8$ and

$$\Gamma_* = \max_{t\in[0,8]} \Gamma(t) \approx 1.51.$$

Thus, the conditions of Theorem 1 are satisfied, and, in this example, it is easy to check that the guaranteed time of capture is $T_g \approx 0.81$.

For the evasion problem, from (2.16) we get $\Upsilon(t) = 1 - 2t$ and $T_o = 0.5$. Therefore, in this example, according to Theorem 2, evasion is possible in the time interval $[0, 0.5)$.

3.2. Example 2

Let in the I-game (1.1)–(1.5), $\varphi(t) = e^{-kt}$ and $\alpha = 4$, $\sigma = 0.25$, $|z_0| = 1$, $k = 0.5$. Here, all the conditions in (1.4) are also satisfied for the function $\varphi(t) = e^{-kt}$. Then from Proposition 1 we have $T_d \approx 5.5$. Hence, by (2.4) and (2.5), we get $\Gamma(t) = t\left(2e^{-0.25t} - 0.5\right)$ and

$$\Gamma_* = \max_{t\in[0,5.5]} \Gamma(t) \approx 1.44,$$

which means that the conditions of Theorem 1 are satisfied. For the pursuit problem, from (2.6) we obtain the guaranteed time of capture $T_g \approx 1.25$.

For the evasion problem, from (2.16) we find that $\Upsilon(t) = 1 - 1.5t$ and $T_o = \frac{2}{3}$, that is, evasion is possible in the time interval $\left[0, \frac{2}{3}\right)$.

3.3. Example 3

Let now $\varphi(t) = \dfrac{1}{2\sqrt{t}+1}$ and $\alpha = 2.25$, $\sigma = 0.25$, $|z_0| = 0.5$. In this example, it proceeds $T_d = 16$ from Proposition 1. Then from (2.4) and (2.5) it follows that $\Gamma(t) = t\left(\sqrt{\dfrac{9}{8\sqrt{t}+4}} - 0.5\right)$ and

$$\Gamma_* = \max_{t\in[0,16]} \Gamma(t) \approx 0.71.$$

Then from (2.6) we have the guaranteed time of capture $T_g \approx 1.75$.

In this example, for the evasion problem from (2.16) we find that $\Upsilon(t) = 0.5 - t$, $T_o = 0.5$, and evasion is possible in the time interval $[0, 0.5)$.

Conclusion

In the present work, the problem of pursuit–evasion is solved in the case where the objects move without inertia and the controls of the players satisfy integral non-stationary constraints, i.e., it is assumed that at each time the players have some additional control resource. This situation makes the problem more attractive and adequate for applied processes. The pursuit is considered possible if the pursuer captures the evader at a finite time and the evasion is considered possible if the evader is not captured by the pursuer. To solve the pursuit problem, a parallel approach strategy is proposed, and a special strategy is proposed for the evasion problem. The results obtained in the article are original and can be further generalized to more general classes of differential games.

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Bahrom T. Samatov, Doctor of Physical and Mathematical Sciences, Professor, Department of Mathematical Analysis, Namangan State University, Uychi str., 316, Namangan, 116019, Uzbekistan, e-mail: samatov57@inbox.ru .

Bakhodirjon I. Juraev, PhD student, Department of Mathematics, Andijan State University, University str., 129, Andijan, 170100, Uzbekistan, e-mail: jbahodirjon@bk.ru .

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