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PERTURBATION METHOD, SUBDIFFERENTIALS OF NONSMOOTH ANALYSIS, AND REGULARIZATION OF THE LAGRANGE MULTIPLIER RULE IN NONLINEAR OPTIMAL CONTROL

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We consider the regularization of classical optimality conditions — the Lagrange principle (LP) and the Pontryagin maximum principle (PMP) — in a regular parametric nonlinear (nonconvex) optimal control problem for a parabolic equation with boundary control and with an operator equality-constraint additively depending on the parameter (perturbation method). The set of admissible controls of the problem and the values of the operator defining the equality-constraint are embedded into the spaces of square-summable functions. The main purpose of the regularized LP and PMP is the stable generation of minimizing approximate solutions (MASs) in the sense of J. Warga in the problem under consideration. The regularized LP and PMP are formulated as existence theorems for MASs consisting of minimals (subminimals) of modified Lagrange functionals whose constructions are direct consequences of the subdifferential properties of a lower semicontinuous and, generally speaking, nonconvex value function as a function of the parameter of the problem. They "overcome" the ill-posedness properties of the LP and PMP, are regularizing algorithms, and serve as a theoretical basis for creating algorithms for the practical solution of an optimization problem.

Keywords: nonlinear optimal control, parabolic equation, operator constraint, perturbation method, subdifferentials of nonsmooth analysis, dual regularization, minimizing sequence, regularizing algorithm, Lagrange principle, Kuhn–Tucker theorem, Pontryagin maximum principle.

REFERENCES

1. Alekseev V.M., Tikhomirov V.M., Fomin S.V. *Optimal Control*. NY: Plenum Press, 1987, 309 p. doi: 10.1007/978-1-4615-7551-1 . Original Russian text published in Alekseev V.M., Tikhomirov V.M., Fomin S.V. *Optimal'noe upravlenie*. Moscow: Nauka Publ., 1979, 432 p.
2. Gamkrelidze R.V. The mathematical work of L.S. Pontryagin. *J. Math. Sci.*, 2000, vol. 100, no. 5, pp. 2447–2457. doi: 10.1007/BF02673835 .
3. Fursikov A.V. *Optimal control of distributed systems: Theory and applications*. Translations of Mathematical Monographs, vol. 187. Providence, RI: AMS, 2000, 305 p. ISBN: 978-0-8218-1382-9 . Original Russian text published in Fursikov A.V. *Optimal'noe upravlenie raspredelennymi sistemami. Teoriya i prilozheniya*, Novosibirsk: Nauchnaya Kniga Publ., 1999, 352 p.
4. Tröltzsch F. *Optimal control of partial differential equations: Theory, methods and applications*. Ser. Graduate Studies in Math., vol. 112. Providence, RI: AMS, 2010, 400 p. doi: 10.1090/gsm/112 .
5. Sumin M.I. Regularized parametric Kuhn–Tucker theorem in a Hilbert space. *Comput. Math. Math. Phys.*, 2011, vol. 51, no. 9, pp. 1489–1509. doi: 10.1134/S0965542511090156 .
6. Sumin M.I. Regularized Lagrange principle and Pontryagin maximum principle in optimal control and in inverse problems. *Trudy Inst. Mat. Mekh. UrO RAN*, 2019, vol. 25, no. 1, pp. 279–296 (in Russian). doi: 10.21538/0134-4889-2019-25-1-279-296 .
7. Vasil'ev F.P. *Metody optimizatsii*: v 2-kh. kn. [Optimization methods: In two vols.] Moscow: MCCME Publ., 2011, 1056 p.
8. Tikhonov A.N., Arsenin V.Ya. *Solutions of ill-posed problems*. Washington: Winston; NY: Halsted Press, 1977, 258 p. ISBN: 0470991240 . Original Russian text published in Tikhonov A.N., Arsenin V.Ya. *Metody resheniya nekorrektnykh zadach*, Moscow: Nauka Publ., 1986, 288 p.
9. Sumin M.I. Parametric dual regularization for an optimal control problem with pointwise state constraints. *Comput. Math. Math. Phys.*, 2009, vol. 49, no. 12, pp. 1987–2005. doi: 10.1134/S096554250912001X .

10. Sumin M.I. Regularization of the Pontryagin maximum principle in a convex optimal boundary control problem for a parabolic equation with an operator equality constraint. *Trudy Inst. Mat. Mekh. UrO RAN*, 2021, vol. 27, no. 2, pp. 221–237 (in Russian). doi: 10.21538/0134-4889-2021-27-2-221-237 .
11. Sumin M.I. Stable sequential Kuhn-Tucker theorem in iterative form or a regularized Uzawa algorithm in a regular nonlinear programming problem. *Comput. Math. Math. Phys.*, 2015, vol. 55, no. 6, pp. 935–961. doi: 10.1134/S0965542515060111 .
12. Loewen P.D. *Optimal control via nonsmooth analysis*. Ser. CRM Proceedings and Lecture Notes, vol. 2. Providence, RI: AMS, 1993, 153 p. doi: 10.1090/crmp/002 .
13. Clarke F.H., Ledyaev Yu.S., Stern R.J., Wolenski P.R. *Nonsmooth analysis and control theory*. Ser. Graduate Texts in Math., vol. 178. NY: Springer-Verlag, 1998, 278 p. doi: 10.1007/b97650 .
14. Mordukhovich B.S. *Variational analysis and generalized differentiation, I: Basic Theory*. Berlin: Springer, 2006, 579 p. doi: 10.1007/3-540-31247-1 .
15. Ladyzhenskaya O.A., Solonnikov V.A., Ural'tseva N.N. *Linear and quasilinear equations of parabolic type*. Providence, RI: AMS, 1968, 648 p. ISBN: 978-0-8218-1573-1 . Original Russian text published in Ladyzhenskaya O.A., Solonnikov V.A., Ural'tseva N.N. *Lineinye i kvazilineinye uravneniya parabolicheskogo tipa*, Moscow: Nauka Publ., 1967, 736 p.
16. Warga J. *Optimal control of differential and functional equations*. NY: Acad. Press, 1972, 531 p. ISBN: 9781483259192 . Translated to Russian under the title *Optimal'noe upravlenie differentsial'nymi i funktsional'nymi uravneniyami*, Moscow: Nauka Publ., 1977, 624 p.
17. Plotnikov V.I. Uniqueness and existence theorems and a priori properties of generalized solutions. *Sov. Math., Dokl.*, 1965, vol. 6, pp. 1405–1407.
18. Casas E., Raymond J.P., Zidani H. Pontryagin's principle for local solutions of control problems with mixed control-state constraints. *SIAM J. Control Optim.*, 2000, vol. 39, no. 4, pp. 1182–1203. doi: 10.1137/S0363012998345627 .
19. Dunford N., Schwartz J.T. *Linear operators,* I*: General theory*. NY: Interscience Publ., 1958, 858 p. ISBN: 0470226056 . Translated to Russian under the title *Lineinye operatory, I: Obshchaya teoriya*, Moscow: Inostr. Lit. Publ., 1962, 896 p.
20. Smirnov V.I. *Kurs vysshei matematiki. Tom 5* [Course of higher mathematics. Vol. 5]. Moscow: GIFML Publ., 1959, 636 p.
21. Trenogin V.A. *Funktsional'nyi analiz* [Functional analysis]. Moscow: Nauka Publ., 1980, 496 p.
22. Minoux M. *Mathematical programming: Theory and algorithms*. NY: Wiley, 1986, 489 p. ISBN: 0471901709 . Translated to Russian under the title *Matematicheskoe programmirovanie: Teoriya i algoritmy*, Moscow: Nauka Publ., 1990, 488 p.
23. Bertsekas D.P. *Constrained optimization and Lagrange multiplier methods*. NY: Academic, 1982, 395 p. doi: 10.1016/C2013-0-10366-2 . Translated to Russian under the title *Uslovnaya optimizatsiya i metody mnozhitelei Lagranzha*, Moscow: Radio i Svyaz' Publ., 1987, 400 p.
24. Gol'shtein E.G., Tret'yakov N.V. *Modified Lagrangians and monotone maps in optimization*. NY: Wiley, 1996, 438 p. ISBN: 0471548219 . Original Russian text published in Gol'shtein E.G., Tret'yakov N.V. *Modifitsirovannye funktsii Lagranzha. Teoriya i metody optimizatsii*, Moscow: Nauka Publ., 1989, 399 p.

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