TRUDY INSTITUTA MATEMATIKI I MEKHANIKI UrO RAN

Vol. 28 No. 3 **2022**

MSC: 93C20, 93C23, 35B30, 47B38

DOI: 10.21538/0134-4889-2022-28-3-188-201

VOLTERRA FUNCTIONAL EQUATIONS IN THE THEORY OF OPTIMIZATION OF DISTRIBUTED SYSTEMS. ON THE PROBLEM OF SINGULARITY OF CONTROLLED INITIAL–BOUNDARY VALUE PROBLEMS

V. I. Sumin

Earlier the author proposed a rather general form of describing controlled *initial–boundary value problems* (IBVPs) by means of *Volterra functional equations* (VFEs)

$$z(t) = f\left(t, A[z](t), v(t)\right), \quad t \equiv \{t^1, \ldots, t^N\} \in \Pi \subset \mathbb{R}^N, \quad z \in L_p^m \equiv (L_p(\Pi))^m,$$

where $f(.,.,.) : \Pi \times \mathbb{R}^l \times \mathbb{R}^s \to \mathbb{R}^m$, $v(.) \in \mathcal{D} \subset L_k^s$ is a control function, and $A : L_p^m(\Pi) \to L_q^l(\Pi)$ is a linear operator that is *Volterra for some system* $\mathbf{T}$ of subsets of Π in the following sense: for any $H \in \mathbf{T}$, the restriction $A[z]|_H$ does not depend on the values of $z|_{\Pi \setminus H}$, $p, q, k \in [1, +\infty]$ (this definition of a Volterra operator is a direct multidimensional generalization of the well-known Tikhonov definition of a functional Volterra type operator). Various IBVPs (for nonlinear hyperbolic and parabolic equations, integro-differential equations, equations with delay, etc.) are reduced by the method of inversion of the main part to such functional equations. The transition to an equivalent VFE-description of IBVPs is adequate to many problems of distributed optimization (obtaining conditions for maintaining the global solvability of equations under perturbed controls, substantiation of numerical methods of optimal control, derivation of necessary optimality conditions, study of singular controls for necessary optimality conditions, etc.). In particular, the author proposed (using such a description) a scheme for obtaining sufficient stability conditions (under perturbations of control) for the existence of global solutions for IBVPs. In the present paper, the effectiveness (for the theory of optimal control) of such a description of IBVPs is demonstrated by an example of a controlled semilinear parabolic equation. The problems of obtaining sufficient conditions for the preservation (under perturbation of control) of the global solvability of IBVPs and the derivation of necessary optimality conditions for singular in the sense of J.-L. Lions optimal control problems are considered. It is shown that some optimization problems that were classified as singular can in fact be classified as nonsingular, since the necessary optimality conditions for them may be derived by bringing the problems to the classical form and varying the controls.

Keywords: Volterra functional equations, controlled initial–boundary value problems, conditions for maintaining global solvability, singular optimality systems.

REFERENCES

1. Dunford N., Schwartz J.T. *Linear operators. I. General theory.* NY: Interscience Publishers, 1958, 858 p. ISBN: 0470226056 . Translated to Russian under the title *Lineinye operatory. Obshchaya teoriya*, Moscow: Inostr. Lit. Publ., 1962, 896 p.
2. Ioffe A.D., Tikhomirov V.M. *Theory of extremal problems.* Amsterdam; NY; Oxford: North-Holland, 1979, 460 p. doi: 10.1016/s0168-2024(09)x7002-5 . Original Russian text published in Ioffe A.D., Tikhomirov V.M. *Teoriya ekstremal'nykh zadach*, Moscow: Nauka Publ., 1974, 480 p.
3. Ladyzenskaja O.A., Solonnikov V.A., Ural'ceva N.N. *Linear and quasi-linear equations of parabolic type.* Providence: AMS, 1968, 648 p. doi: 10.1090/mmono/023 . Original Russian text published in Ladyzhenskaya O.A., Solonnikov V.A., Ural'tseva N.N. *Lineinye i kvazilineinye uravneniya parabolicheskogo tipa*, Moscow: Nauka Publ., 1967, 736 p.
4. Lions J. *Control of distributed singular systems.* Paris: Gauthier-Villars, 1985, 552 p. ISBN: 9782040157487 . Translated to Russian under the title *Upravlenie singulyarnymi raspredelennymi sistemami*, Moscow: Nauka Publ., 1987, 368 p.
5. Sumin V.I. Stability of the existence of a global solution to the first boundary value problem for a controllable parabolic equation. *Differ. Uravn.*, 1986, vol. 22, no. 9, pp. 1587–1595 (in Russian).

6. Sumin V.I. On the problem of singularity of controllable distributed parameter systems. III. *Vestn. Nizhegorod. Univ., Mat. Model. Optim. Upr.*, 2003, vol. 1(25), pp. 164–174 (in Russian).
7. Sumin V.I. Controlled Volterra functional equations and the contraction mapping principle. *Trudy Instituta Matematiki i Mekhaniki URO RAN*, 2019, vol. 26, no. 1, pp. 262–278 (in Russian). doi: 10.21538/0134-4889-2019-25-1-262-278 .
8. Sumin V.I. Volterra functional equations in the stability problem for the existence of global solutions of distributed controlled systems. *Vestn. Ross. Univ. Matem.*, 2020, vol. 25, no. 132, pp. 422–440 (in Russian). doi: 10.20310/2686-9667-2020-25-132-422-440 .
9. Tikhonov A.N. On functional equations of Volterra type and their applications to certain problems of mathematical physics. *Byulleten' MGU. Sektsiya A*, 1938, vol. 1, no. 8, pp. 1–25 (in Russian).
10. Fursikov A.V. *Optimal control of distributed systems: Theory and applications.* Providence: AMS, 2000, 305 p. doi: 10.1090/MMONO/187187 . Original Russian text published in Fursikov A.V. *Optimal'noe upravlenie raspredelennymi sistemami: Teoriya prilozheniya*, Novosibirsk: Nauchnaya kniga Publ., 1999, 352 p.
11. Chernov A.V. On overcoming the singularity of distributed control systems. In: *Proceedings of the Third All-Russian Scientific Conference (29–31 May 2006) "Matematicheskoe modelirovanie i kraevye zadachi"*, part 2. Samara: Samara State Technical Univ., 2006, pp. 171–174 (in Russian).
12. Barbu V. Necessary conditions for distributed control problems governed by parabolic variational inequalities. *SIAM J. Control Optim.*, 1981, vol. 19, no. 1, pp. 64–86. doi: 10.1137/0319006 .
13. Fursikov A. Lagrange principle for problems of optimal control of ill-posed or singular distributed systems. *J. Math. Pures Appl.*, 1992, vol. 71, no. 2, pp. 139–195.
14. Rota G.-C., Strang W.G. A note on the joint spectral radius. *Indag. Math.*, 1960, vol. 63, pp. 379–381. doi: 10.1016/S1385-7258(60)50046-1 .
15. Shulman V.S., Turovskii Yu.V. Joint spectral radius, operator semigroups, and a problem of W. Wojtynski. *J. Funct. Anal.*, 2000, vol. 177, no. 2, pp. 383–441. doi: 10.1006/jfan.2000.3640 .
16. Sumin V. Volterra functional-operator equations in the theory of optimal control of distributed systems. *IFAC-PapersOnLine*, 2018, vol. 51, no. 32, pp. 759–764. doi: 10.1016/j.ifacol.2018.11.454 .

Received June 15, 2022
Revised July 15, 2022
Accepted July 18, 2022

Funding Agency: This work was supported by the Russian Foundation for Basic Research (project no. 20-01-00199_a).

Vladimir Iosifovich Sumin, Dr. Phys.-Math. Sci., Prof., Nizhny Novgorod State University named after N. I. Lobachevsky, Nizhny Novgorod, 603950 Russia; Derzhavin Tambov State University, Tambov, 392000 Russia, e-mail: v_sumin@mail.ru .

Cite this article as: V. I. Sumin. Volterra functional equations in the theory of optimization of distributed systems. On the problem of singularity of controlled initial–boundary value problems. *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2022, vol. 28, no. 3, pp. 188–201.