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ON A CONDITION FOR THE COINCIDENCE OF TRANSFORM SPACES FOR FUNCTIONALS IN A HILBERT SPACE

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The paper considers the following problem. Let H be some reproducing kernel Hilbert space consisting of functions given on a set $\Omega \subset \mathbb{C}^n$, $n \geq 1$, and let $\{e_1(\cdot, \xi)\}_{\xi \in \Omega_1}$ and $\{e_2(\cdot, \xi)\}_{\xi \in \Omega_1}$ be some complete systems of functions in H , where $\Omega_1 \subset \mathbb{C}^m$, $m \geq 1$. Define

$$\begin{aligned}\tilde{f}(z) &\stackrel{\text{def}}{=} (e_1(\cdot, z), f)_H \quad \forall z \in \Omega_1, \quad \tilde{H} = \{\tilde{f}, f \in H\}, \\ (\tilde{f}_1, \tilde{f}_2)_{\tilde{H}} &\stackrel{\text{def}}{=} (f_2, f_1)_H, \quad \|\tilde{f}_1\|_{\tilde{H}} = \|f_1\|_H \quad \forall \tilde{f}_1, \tilde{f}_2 \in \tilde{H}, \\ \hat{f}(z) &\stackrel{\text{def}}{=} (e_2(\cdot, z), f)_H \quad \forall z \in \Omega_1, \quad \hat{H} = \{\hat{f}, f \in H\}, \\ (\hat{f}_1, \hat{f}_2)_{\hat{H}} &\stackrel{\text{def}}{=} (f_2, f_1)_H, \quad \|\hat{f}_1\|_{\hat{H}} = \|f_1\|_H \quad \forall \hat{f}_1, \hat{f}_2 \in \hat{H}.\end{aligned}$$

It is required to find a condition under which the spaces \hat{H} and \tilde{H} coincide, i.e., \hat{H} and \tilde{H} consist of the same functions and

$$\|f\|_{\hat{H}} = \|f\|_{\tilde{H}} \quad \forall f \in \hat{H} = \tilde{H}.$$

We also study the question of conditions under which the spaces \hat{H} and \tilde{H} are equivalent. In the case when the systems of functions $\{e_j(\cdot, \xi)\}_{\xi \in \Omega_1}$, $j = 1, 2$, are orthosimilar decomposition systems in the space H with the same measure μ given on Ω_1 , a criterion is established; more exactly, a condition is found that is necessary and sufficient for the coincidence (equivalence) of the spaces \hat{H} and \tilde{H} . Note that, in the case of an arbitrary space H and arbitrary systems of functions $\{e_1(\cdot, \xi)\}_{\xi \in \Omega_1}$ and $\{e_2(\cdot, \xi)\}_{\xi \in \Omega_1}$ that are complete in H , the found condition is always necessary; i.e., if the spaces \hat{H} and \tilde{H} coincide (are equivalent), then this condition is fulfilled. In the case when the systems of functions $\{e_1(\cdot, \xi)\}_{\xi \in \Omega_1}$ and $\{e_2(\cdot, \xi)\}_{\xi \in \Omega_1}$ are orthosimilar decomposition systems in the space H with different measures μ_1 and μ_2 , respectively, given on Ω_1 , we construct specific examples of spaces H and systems of functions $\{e_1(\cdot, \xi)\}_{\xi \in \Omega_1}$ and $\{e_2(\cdot, \xi)\}_{\xi \in \Omega_1}$ complete in H and such that the specified condition is met, but the spaces \hat{H} and \tilde{H} are not the same (not equivalent).

Keywords: orthosimilar decomposition systems, reproducing kernel Hilbert space, Riesz basis, problem of describing the dual space.

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