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EXACT SOLUTIONS OF DIFFUSION WAVE TYPE FOR A NONLINEAR SECOND-ORDER PARABOLIC EQUATION WITH DEGENERATION

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The paper deals with a nonlinear evolutionary second-order parabolic equation with degeneration, which is a mathematical model for a number of physical and biological processes. We consider the problem of constructing and exploring exact solutions having the type of diffusion (heat, filtration) wave with a specified front. By applying a special kind of ansatz, their construction reduces to the integration of the Cauchy problem for an ordinary differential equation, which inherits the singularity of the original formulation. A three-stage approach is proposed to eliminate the singularity. At the first stage, the order of the equation is reduced by passing to the phase plane. Next, a solution is constructed in the form of a series in powers of a new independent variable, which previously was the original unknown function. Finally, the convergence of the series is proved by constructing a positive majorant. A special section is devoted to finding a constructive estimate of the convergence radius of the series. This estimate, in particular, shows that the radius is considerably different from zero. The proposed approach to the construction of estimates is highly adaptive, which allows us to improve the obtained estimates significantly if the input constants are specified.

Keywords: nonlinear parabolic equation, diffusion wave, exact solutions, traveling wave, series, convergence.

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