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ON A CLASS OF VERTEX-PRIMITIVE ARC-TRANSITIVE AMPLY REGULAR GRAPHS [1,2]

M. P. Golubyatnikov, N. V. Maslova

A simple k-regular graph with v vertices is an amply regular graph with parameters (v, k, λ, μ) if any two adjacent vertices have exactly λ common neighbors and any two vertices which are at distance 2 in this graph have exactly μ common neighbors. Let G be a finite group, $H \leq G$, $\mathfrak{H} = \{H^g \,|\, g \in G\}$ be the corresponding conjugacy class of subgroups of G, and $1 \leq d$ be an integer. We construct a simple graph $\Gamma(G, H, d)$ as follows. The vertices of $\Gamma(G, H, d)$ are the elements of $\mathfrak{H}$, and two vertices H_1 and H_2 from $\mathfrak{H}$ are adjacent in $\Gamma(G, H, d)$ if and only if $|H_1 \cap H_2| = d$. In this paper we prove that if q is a prime power with $13 \leq q \equiv 1 \pmod 4$, $G = SL_2(q)$, and H is a dihedral maximal subgroup of G of order $2(q-1)$, then the graph $\Gamma(G, H, 8)$ is a vertex-primitive arc-transitive amply regular graph with parameters $\left(\dfrac{q(q+1)}{2}, \dfrac{q-1}{2}, 1, 1\right)$ and with $\mathrm{Aut}(PSL_2(q)) \leq \mathrm{Aut}(\Gamma)$. Moreover, we prove that $\Gamma(G, H, 8)$ has a perfect 1-code, in particular, its diameter is more than 2.

Keywords: finite simple group, arc-transitive graph, amply regular graph, edge-regular graph, graph of girth 3, Deza graph, perfect 1-code.

М. П. Голубятников, Н. В. Маслова. О классе вершинно-примитивных транзитивных на дугах вполне регулярных графов.

Обыкновенный k-регулярный граф с v вершинами называется вполне регулярным с параметрами (v, k, λ, μ), если любые две смежные вершины имеют точно λ общих соседей, а любые вершины, находящиеся на расстоянии 2 в этом графе, имеют точно μ общих соседей. Пусть G — конечная группа, $H \leq G$, $\mathfrak{H} = \{H^g \,|\, g \in G\}$ — соответствующий класс сопряженности подгрупп группы G и $1 \leq d$ — целое число. Построим обыкновенный граф $\Gamma(G, H, d)$ следующим образом: вершинами графа $\Gamma(G, H, d)$ являются элементы класса $\mathfrak{H}$, и две различные вершины H_1 и H_2 из $\mathfrak{H}$ смежны в $\Gamma(G, H, d)$ тогда и только тогда, когда $|H_1 \cap H_2| = d$. В данной работе мы доказываем, если q — степень простого числа такая, что $13 \leq q \equiv 1 \pmod 4$, $G = SL_2(q)$ и H — диэдральная максимальная подгруппа группы G порядка $2(q-1)$, то граф $\Gamma = \Gamma(G, H, 8)$ является вершинно примитивным транзитивным на дугах вполне регулярным графом с параметрами $\left(\dfrac{q(q+1)}{2}, \dfrac{q-1}{2}, 1, 1\right)$, при этом $\mathrm{Aut}(PSL_2(q)) \leq \mathrm{Aut}(\Gamma)$. Более того, мы показываем, что $\Gamma = \Gamma(G, H, 8)$ содержит совершенный 1-код, в частности, диаметр этого графа больше 2.

Ключевые слова: конечная простая группа, транзитивный на дугах граф, вполне регулярный граф, реберно регулярный граф, граф обхвата 3, граф Деза, совершенный 1-код.

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Throughout the paper we consider only finite groups and simple graphs, and henceforth the term group means finite group, and the term graph means simple graph (undirected graph without loops and multiple edges). Our terminology and notation are mostly standard and can be found, for example, in [2; 3; 5].

A simple k-regular graph with v vertices is an *amply regular graph* with parameters (v, k, λ, μ) if any two adjacent vertices have exactly λ common neighbors and any two vertices which are at distance 2 in this graph have exactly μ common neighbors. A class of amply regular graphs with $\lambda = \mu$ is of a special interest; such graphs for $\lambda = \mu \geq 2$ have been studied by M. Mulder [11]. But

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there are no many results about amply regular graphs with $\lambda = \mu = 1$. At the same time, some of such graphs due to their properties probably can be used in the architecture of multiprocessor systems similarly to the Hypercubes and the Star graphs [1; 7; 12].

Let G be a group, $H \le G$, $\mathfrak{H} = \{H^g \mid g \in G\}$ be the corresponding conjugacy class of subgroups of G, and $d \ge 1$ be an integer. We construct a graph $\Gamma(G, H, d)$ as follows. The vertices of $\Gamma(G, H, d)$ are the elements of $\mathfrak{H}$, and two vertices H_1 and H_2 from $\mathfrak{H}$ are adjacent in $\Gamma(G, H, d)$ if and only if $|H_1 \cap H_2| = d$.

Let Γ be a graph. Recall that a subset C of vertices of Γ is a 1-*perfect code* in Γ if the vertex set of Γ is a disjoint union of balls of radius 1 with centers at the vertices from C.

In this paper we prove the following theorem.

Theorem 1. *Let $G = SL_2(q)$, where q is a prime power with $13 \le q \equiv 1 \pmod 4$, H be a maximal dihedral subgroup of G of order $2(q-1)$,*[3] *and $\Gamma = \Gamma(G, H, 8)$. Then the following statements hold*:

(i) Γ *is a vertex-primitive arc-transitive amply regular graph with parameters*

$$\Big(\frac{q(q+1)}{2}, \frac{q-1}{2}, 1, 1\Big);$$

(ii) $\mathrm{Aut}(PSL_2(q)) \le \mathrm{Aut}(\Gamma)$;

(iii) Γ *has a perfect* 1-*code, in particular,* Γ *is of diameter more than* 2.

A simple graph is called a *Deza graph* with parameters (v, k, b, a) if it has v vertices, is regular of valency k and any two different vertices have either a or b common neighbors in this graph.

Corollary 1. *Under the notation system of Theorem* 1*, the graph* Γ *is a Deza graph of girth* 3 *with parameters*

$$\Big(\frac{q(q+1)}{2}, \frac{q-1}{2}, 1, 0\Big).$$

L. Lovasz [8, Problem 11] asked whether every finite connected vertex-transitive graph has a Hamilton path. Our calculations with GAP [4] give that if $q \le 41$, then the graph $\Gamma(G, H, 8)$ has a Hamiltonian cycle. Thus, we have the following hypothesis:

Hypothesis 1. *For each q, the graph $\Gamma(G, H, 8)$ has a Hamiltonian cycle.*

In 2009, D. Marusic and R. Scapellato [9] constructed a class of vertex-transitive non-Cayley graphs as orbital graphs with $PSL_2(p)$, where p is a prime, acting by right multiplication on the right cosets of a dihedral subgroup of order $p - 1$. Really, our graphs $\Gamma(G, H, 8)$ are orbital graphs with $PGL_2(q)$ acting by right multiplication on the right cosets of a dihedral subgroup of order $q - 1$, and our calculations with GAP [4] give that if $q \le 41$, then $\Gamma(G, H, 8)$ is a non-Cayley vertex-transitive graph. Thus, we have the following hypothesis:

Hypothesis 2. *For each q, the graph $\Gamma(G, H, 8)$ is a non-Cayley vertex-transitive graph.*

1. Preliminaries

The following easily proved assertions give some information about automorphism groups of the graphs $\Gamma(G, H, d)$ for arbitrary G, H, and d.

[3] If $q \ge 13$, then $SL_2(q)$ has a maximal dihedral subgroup of order $2(q-1)$. A detailed description of this subgroup can be found in Lemma 1 below.

Proposition 1. *Let G be a finite group, $\overline{} : G \to \mathrm{Aut}(G)$ be the natural homomorphism, $H \le G$, $\mathfrak{H} = \{H^g \mid g \in G\}$ be the corresponding conjugacy class of subgroups of G, and $B = \overline{G}N_{\mathrm{Aut}(G)}(H)$. Then the following statements hold*:

(i) *there exists a homomorphism $\psi : B \to \mathrm{Aut}(\Gamma(G,H,d))$ such that $\psi(\overline{G})$ is a normal subgroup of $\psi(B)$,*
$$\psi(B) \cong B/(\cap_{H_i\in\mathfrak{H}} N_B(H_i)) \text{ and } \psi(\overline{G}) \cong G/(\cap_{H_i\in\mathfrak{H}} N_G(H_i));$$

(ii) *if $Z(G) \le H$, then $G/\cap_{H_i\in\mathfrak{H}} N_G(H_i) \cong \overline{G}/(\cap_{H_i\in\mathfrak{H}} N_{\overline{G}}(\overline{H_i})))$ and $B/(\cap_{H_i\in\mathfrak{H}} N_B(H_i))) = B/(\cap_{H_i\in\mathfrak{H}} N_B(\overline{H_i})))$.*

Proof. The group $\overline{G} = G/Z(G)$ acts on $\mathfrak{H}$ in the same way as G, and it is well-known that $\overline{G} = \mathrm{Inn(G)}$ is a normal subgroup of $A = \mathrm{Aut}(G)$; therefore, $\overline{G} \trianglelefteq B$. Moreover, it is clear that the group $B = \overline{G}N_A(H)$ acts on $\mathfrak{H}$. If $b \in B$, then
$$|H \cap H^g| = |H^b \cap H^{gb}|;$$
therefore, there is a homomorphism ψ from B to $\mathrm{Aut}(\Gamma(G,H,d))$ with $\ker(\psi) = \cap_{H_i\in\mathfrak{H}} N_B(H_i)$. Further, $\overline{} \circ \psi : G \to \mathrm{Aut}(\Gamma(G,H,d))$ is a homomorphism, and it is clear that
$$\ker(\overline{} \circ \psi) = \cap_{H_i\in\mathfrak{H}} N_G(H_i).$$
Now Statement (i) follows from the fundamental theorem on homomorphisms.

If $Z(G) \le H$, then for each $H_i \in \mathfrak{H}$, $Z(G) \le H_i$. Thus, $Z(G) \le \cap_{H_i\in\mathfrak{H}} N_G(H_i)$ and $N_G(H_i) = N_G(H_iZ(G))$. Therefore, by the fundamental theorem on homomorphisms, we have $N_G(H_i)/Z(G) = N_G(H_iZ(G))/Z(G) \cong N_{\overline{G}}(\overline{H_i})$ and
$$G/\cap_{H_i\in\mathfrak{H}} N_G(H_i) \cong (G/Z(G))/((\cap_{H_i\in\mathfrak{H}} N_G(H_i))/Z(G))$$
$$= (G/Z(G))/(\cap_{H_i\in\mathfrak{H}}(N_G(H_i))/Z(G))) \cong \overline{G}/(\cap_{H_i\in\mathfrak{H}} N_{\overline{G}}(\overline{H_i})).$$
Similarly, we have $Z(G) \le H_i$ and $N_B(H_i) = N_B(H_iZ(G)) = N_B(\overline{H_i})$ for each $H_i \in \mathfrak{H}$. Thus, Statement (ii) holds. □

Remark 1. The condition $Z(G) \le H$ in Proposition 1(ii) is essential. Let
$$G = \langle a, b \mid a^4 = 1, b^2 = 1, bab = a^{-1}\rangle$$
be a dihedral group of order 8 and $H = \langle b\rangle$. Then $Z(G) = \langle a^2\rangle$, $N_G(H) = \langle a^2, b\rangle = HZ(G)$, and $N_G(HZ(G)) = G$. Thus, $|N_G(H)/Z(G)| = 2$, $N_{G/Z(G)}((Z(G)\langle b\rangle)/Z(G)) = G/Z(G)$, and $|G/Z(G)| = 4$.

Remark 2. Both subgroups $\psi(B)$ and $\psi(\overline{G})$ can be proper non-normal subgroups of the group $\mathrm{Aut}(\Gamma(G,H,d))$. For example, if d does not divide $|H|$, then the graph $\Gamma(G,H,d)$ is a coclique and $\mathrm{Aut}(\Gamma(G,H,d)) = Sym(\mathfrak{H})$.

Corollary 2. *The graph $\Gamma(G,H,d)$ is vertex-transitive for all G, H, and d.*

Proof. It is clear that the group $\psi(\overline{G})$ acts transitively on the vertices of $\Gamma(G,H,d)$ for all G, H, and d by definition of $\Gamma(G,H,d)$. Thus, the corollary holds. □

Now we introduce a notation system, which will be valid until the end of the paper. Fix a prime p and an integer $m > 0$. Let $GF(q)$ be the finite field of order $q = p^m$ and $G = SL_2(q)$ be the corresponding special linear group, i.e., the group of 2×2 invertible matrices over $GF(q)$ with determinant 1. Let H be a dihedral maximal subgroup of G of order $2(q-1)$ and $\Gamma = \Gamma(G,H,8)$. For any vertex v of Γ, denote by $\Gamma(v)$ the subgraph of Γ induced by the set of all vertices which are adjacent to v in Γ.

Lemma 1. *If $q \ge 13$, then*

(1) *G has a maximal dihedral subgroup H of order $2(q-1)$; moreover, $N_G(H) = H$.*

(2) *The following statements are equivalent:*

  (i) $q \equiv 1 \pmod 4$;

  (ii) -1 *is a square in* $GF(q)$;

  (iii) $|G : H|$ *is odd.*

(3) *H is the stabilizer in G of a decomposition of the natural 2-dimensional module V of G into a direct sum of two subspaces of dimension 1; moreover, in a suitable basis of V,*

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;\Big|\; a \in GF(q)^* \right\} \bigcup \left\{ \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} \;\Big|\; a \in GF(q)^* \right\}.$$

(4) *There are subgroups K and L of $\mathrm{Aut}(\Gamma)$ with the following properties: $K \trianglelefteq L$, $K \cong PSL_2(q)$, $L \cong \mathrm{Aut}(PSL_2(q))$, and K is vertex-primitive on Γ. In particular, Γ is connected.*

Proof. Statement (1) follows, for example, from [2, Tables 8.1, 8.3].

Statements (2)(i) and (2)(ii) are equivalent, for example, by [6, P. 42, Corollary 2]; Statements (2)(i) and (2)(iii) are equivalent, for example, by [10].

Statement (3) follows from [2, Tables 8.1, 8.3, and § 2.2.2].

Prove Statement (4). Let $\overline{G} = G/Z(G) \cong PSL_2(q)$. First of all, note that $Z(G) \leq H$. By [2, Tables 8.1, 8.3], $\mathrm{Aut}(\overline{G}) = \mathrm{Inn}(\overline{G})N_{\mathrm{Aut}(\overline{G})}(\overline{H})$. Now by Proposition 1, the group $\mathrm{Aut}(\Gamma)$ contains subgroups $K \cong \overline{G}$ and $L \cong \mathrm{Aut}(\overline{G})$ with $K \trianglelefteq L$. Moreover, $\mathrm{Stab}_K(H) \cong \overline{H}$ is a maximal subgroup in K; therefore, K is vertex-primitive on Γ; in particular, Γ is connected. □

Further we assume that $q \equiv 1 \pmod 4$, and by ξ we denote an element of the field $GF(q)$ with the property $\xi^2 = -1$. Moreover, we fix a basis of the natural 2-dimensional module of G such that in this basis H has a shape as in Statement (3) of Lemma 1, and further we assume that all the matrices are presented in this basis.

Let $a \in GF(q)^*$. Put

$$R_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad S_a = \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}.$$

It is clear that the following equalities hold:

$$R_a \cdot R_b = R_{ab}, \quad S_a \cdot S_b = R_{-ab^{-1}},$$
$$R_a \cdot S_b = S_{ab}, \quad \text{and} \quad S_a \cdot R_b = S_{ab^{-1}}$$

Lemma 2. *If $P \in G \setminus H$, then the following statements hold:*

(1) *$PR_aP^{-1} = R_b$ if and only if $a = b$ and $a^2 = 1$;*

(2) *$PR_aP^{-1} = S_b$ if and only if $a^2 = -1$ and* $P = \begin{pmatrix} x\dfrac{b}{a} & -\dfrac{1}{2x} \\ x & \dfrac{a}{2xb} \end{pmatrix}$ *for some $x \in GF(q)^*$;*

(3) *$PS_aP^{-1} = R_b$ if and only if $b^2 = -1$ and* $P = \begin{pmatrix} \dfrac{x}{ab} & -x \\ \dfrac{1}{2x} & \dfrac{ab}{2x} \end{pmatrix}$ *for some $x \in GF(q)^*$;*

(4) *$PS_aP^{-1} = S_b$ if and only if* $P = \begin{pmatrix} xb & -aby \\ y & xa \end{pmatrix}$ *for some $x, y \in GF(q)^*$ satisfying the equality $x^2 + y^2 = (ab)^{-1}$. Moreover, for a fixed matrix P of this form, the equality $PS_{a'}P^{-1} = S_{b'}$ holds if and only if $(a', b') = (a, b)$ or $(a', b') = (-a, -b)$.*

Proof. Let $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$.

Prove Statement (1). We have

$$PR_a - R_bP = \begin{pmatrix} p_{11}(a-b) & -p_{12}\dfrac{ab-1}{a} \\ p_{21}\dfrac{ab-1}{b} & p_{22}\dfrac{b-a}{ab} \end{pmatrix} = 0.$$

Hence $PR_aP^{-1} = R_b$ if and only if $a = b$ and $a^2 = 1$. Thus, Statement (1) holds.

Prove Statement (2). Consider the matrix equation

$$0 = PR_a - S_bP = \begin{pmatrix} ap_{11} - bp_{21} & a^{-1}p_{12} - bp_{22} \\ b^{-1}p_{11} + ap_{21} & b^{-1}p_{12} + a^{-1}p_{22} \end{pmatrix}. \tag{1.1}$$

Matrix equation (1.1) can be considered as a system of linear equations with respect to variables p_{11}, p_{12}, p_{21}, and p_{22} with matrix

$$M = \begin{pmatrix} a & 0 & -b & 0 \\ 0 & a^{-1} & 0 & -b \\ b^{-1} & 0 & a & 0 \\ 0 & b^{-1} & 0 & a^{-1} \end{pmatrix} \text{ such that } \det M = \frac{(a^2+1)^2}{a^2}. \tag{1.2}$$

It is clear that matrix equation (1.1) has a non-trivial solution if and only if $\det M = 0$, i. e., $a = \pm\xi$. Note that if $a = \pm\xi$, then the rank of the matrix M in (1.2) is 2 and p_{12} and p_{21} can be chosen as independent variables. Solving equation (1.1) for p_{11} and p_{22}, we get

$$p_{11} = \frac{p_{21}b}{a} \quad \text{and} \quad p_{22} = -\frac{p_{12}a}{b}.$$

Hence

$$P = \begin{pmatrix} \dfrac{p_{21}b}{a} & p_{12} \\ p_{21} & -\dfrac{p_{12}a}{b} \end{pmatrix} \text{ and } \det P = -2p_{12}p_{21} = 1.$$

Replacing p_{21} by x and substituting $p_{12} = -\dfrac{1}{2x}$, we obtain Statement (2).

Prove Statement (3). Consider the matrix equation

$$0 = PS_a - R_bP = \begin{pmatrix} -bp_{11} - a^{-1}p_{12} & ap_{11} - bp_{12} \\ -b^{-1}p_{21} - a^{-1}p_{22} & ap_{21} - b^{-1}p_{22} \end{pmatrix}. \tag{1.3}$$

Matrix equation (1.3) can be considered as a system of linear equations with respect to variables p_{11}, p_{12}, p_{21}, and p_{22} with matrix

$$M = \begin{pmatrix} -b & -a^{-1} & 0 & 0 \\ a & -b & 0 & 0 \\ 0 & 0 & -b^{-1} & -a^{-1} \\ 0 & 0 & a & -b^{-1} \end{pmatrix} \text{ such that } \det M = \frac{(b^2+1)^2}{b^2}. \tag{1.4}$$

It is clear that matrix equation (1.3) has a non-trivial solution if and only if $\det M = 0$, i. e., $b = \pm\xi$. Note that if $b = \pm\xi$, then the rank of the matrix M in (1.4) is 2 and p_{12} and p_{21} can be chosen as independent variables. Solving equation (1.3) for p_{11} and p_{22}, we get

$$p_{11} = p_{12}\frac{b}{a} \quad \text{and} \quad p_{22} = abp_{21}.$$

Hence

$$P = \begin{pmatrix} p_{12}\dfrac{b}{a} & p_{12} \\ p_{21} & abp_{21} \end{pmatrix} \text{ and } \det P = -2p_{12}p_{21} = 1.$$

Replacing p_{12} by $-x$ and substituting $p_{21} = \dfrac{1}{2x}$, we obtain Statement (3).

Prove Statement (4). Consider the matrix equation

$$0 = PS_a - S_bP = \begin{pmatrix} -\dfrac{p_{12}}{a} - bp_{21} & ap_{11} - bp_{22} \\ \dfrac{p_{11}}{b} - \dfrac{p_{22}}{a} & \dfrac{p_{12}}{b} + ap_{21} \end{pmatrix}. \tag{1.5}$$

Matrix equation (1.5) can be considered as a system of linear equations with respect to variables p_{11}, p_{12}, p_{21}, and p_{22} with matrix

$$M = \begin{pmatrix} 0 & -a^{-1} & -b & 0 \\ a & 0 & 0 & -b \\ b^{-1} & 0 & 0 & -a^{-1} \\ 0 & b^{-1} & a & 0 \end{pmatrix} \text{ such that } \det M = 0.$$

Note that the rank of the matrix M is 2 and p_{22} and p_{21} can be chosen as independent variables. Solving equation (1.5) for p_{11} and p_{12} and replacing p_{22} by xa and p_{21} by y, we get

$$P = \begin{pmatrix} xb & -aby \\ y & xa \end{pmatrix} \text{ and } \det P = ab(x^2 + y^2).$$

Now let $PS_{a'} - S_{b'}P = 0$ for some $a' \neq 0$ and $b' \neq 0$. We get

$$PS_{a'} - S_{b'}P = \begin{pmatrix} \dfrac{(ab - a'b')y}{a'} & (a'b - ab')x \\ \dfrac{(a'b - ab')x}{a'b'} & \dfrac{(a'b' - ab)y}{b'} \end{pmatrix} = 0.$$

Note that $x \neq 0$ and $y \neq 0$; hence, $ab = a'b'$ and $a'b = ab'$. Therefore, $(a', b') = (a, b)$ or $(a', b') = (-a, -b)$. Statement (4) is proved. □

2. Neighborhood structure of the vertex H in Γ

Let

$$\mathcal{R} = \left\{ P_{\alpha,\beta} = \begin{pmatrix} \alpha & -\dfrac{1}{2\beta} \\ \beta & \dfrac{1}{2\alpha} \end{pmatrix} \,\middle|\, \alpha, \beta \in GF(q)^* \right\}$$

be the set of matrices form Statement (2) of Lemma 2. Note that if $P_{\alpha,\beta} \in \mathcal{R}$, then the following equalities hold:

$$PR_\xi P^{-1} = S_{\frac{\alpha}{\beta}\xi} \text{ and } PR_{-\xi}P^{-1} = S_{-\frac{\alpha}{\beta}\xi}.$$

Lemma 3. *If $P \in G$, then $|H \cap H^P| \in \{2(q-1), 8, 4, 2\}$. Moreover, $|H \cap H^P| = 8$ if and only if $P \in \mathcal{R}$.*

Proof. Let $A = H \cap H^P$. If $P \in H$, then $A = H$ and $|A| = 2(q-1)$, as required. If $P \in G \setminus H$ and $|A| = 2(q-1) = |H|$, then $P \in N_G(H)$. But by Statement (1) of Lemma 1, $N_G(H) = H$, a contradiction. Thus, if $P \in G \setminus H$, then $|A| < 2(q-1)$.

Note that by Statement (3) of Lemm 1,

$$H = \{R_a \mid a \in GF(q)^*\} \cup \{S_a \mid a \in GF(q)^*\}.$$

Let $P \in \mathcal{R}$. From Statements (1) and (2) of Lemma 2, there are precisely four matrices R_a for which $R_a^P \in H$, namely:

$$(R_1, R_{-1}, R_\xi, R_{-\xi})^P = (R_1, R_{-1}, S_{\frac{\alpha}{\beta}\xi}, S_{-\frac{\alpha}{\beta}\xi}).$$

Note that

$$\begin{pmatrix} \alpha & -\dfrac{1}{2\beta} \\ \beta & \dfrac{1}{2\alpha} \end{pmatrix} = \begin{pmatrix} xb & -aby \\ y & xa \end{pmatrix},$$

for example, for $(x, y, a, b) = \left(\beta, \beta, \dfrac{1}{2\alpha\beta}, \dfrac{\alpha}{\beta}\right)$. Now from Statement (4) of Lemma 2 we have

$$S_{\frac{1}{2\alpha\beta}}{}^P = S_{\frac{\alpha}{\beta}} \quad \text{and} \quad S_{-\frac{1}{2\alpha\beta}}{}^P = S_{-\frac{\alpha}{\beta}},$$

and there are no $u \in GF(q)^* \setminus \left\{\dfrac{1}{2\alpha\beta}, -\dfrac{1}{2\alpha\beta}\right\}$ with $S_u^P = S_v$ for some $v \in GF(q)^*$.

Note that

$$\begin{pmatrix} \alpha & -\dfrac{1}{2\beta} \\ \beta & \dfrac{1}{2\alpha} \end{pmatrix} = \begin{pmatrix} \dfrac{x}{ab} & -x \\ \dfrac{1}{2x} & \dfrac{ab}{2x} \end{pmatrix},$$

for example, for $(x, a, b) = \left(\dfrac{1}{2\beta}, \dfrac{1}{2\alpha\beta\xi}, \xi\right)$. Now from Statement (3) of Lemma 2 we have

$$S_{\frac{1}{2\alpha\beta\xi}}{}^P = R_\xi \quad \text{and} \quad S_{-\frac{1}{2\alpha\beta\xi}}{}^P = R_{-\xi},$$

and there are no $u \in GF(q)^* \setminus \left\{\dfrac{1}{2\alpha\beta\xi}, -\dfrac{1}{2\alpha\beta\xi}\right\}$ with $S_u^P = R_v$ for some $v \in GF(q)^*$.

Thus, if $P \in \mathcal{R}$, then $|A| = 8$ and

$$A = \left\{R_1, R_{-1}, R_\xi, R_{-\xi}, S_{\frac{\alpha}{\beta}\xi}, S_{-\frac{\alpha}{\beta}\xi}, S_{\frac{\alpha}{\beta}}, S_{-\frac{\alpha}{\beta}}\right\}.$$

Let $P \in G \setminus (H \cup \mathcal{R})$. Assume that P is of the same shape as in Statement (3) of Lemma 2, i. e.,

$$P = \begin{pmatrix} \dfrac{x}{ab} & -x \\ \dfrac{1}{2x} & \dfrac{ab}{2x} \end{pmatrix} \text{ for some } x, a, \text{ and } b \text{ from } GF(q)^*.$$

Put $y = \dfrac{1}{2x}$ and $\dfrac{t}{z} = \dfrac{2x^2}{ab}$, where y, z, and t are from $GF(q)^*$. Then

$$P = \begin{pmatrix} \dfrac{yt}{z} & -\dfrac{1}{2y} \\ y & \dfrac{z}{2ty} \end{pmatrix} \in \mathcal{R};$$

we obtain a contradiction. Thus, by Statements (2) and (3) of Lemma 2, if $P \notin G \setminus (H \cup \mathcal{R})$, then conjugation by P cannot transform a matrix of the form S_u, where $u \in GF(q)^*$, into a matrix of the form R_v, where $v \in GF(q)^*$, and vice versa; therefore, in this case, $|A| \in \{2, 4\}$ by Statements (1) and (4) of Lemma 2. □

Let $\sim$ be an equivalence relation on the set of matrices $\mathcal{R}$ defined as follows:

$$P_1 \sim P_2 \Leftrightarrow P_2^{-1} P_1 \in H.$$

Lemma 4. *We have*
$$|\Gamma(H)| = |\mathcal{R}/\sim| = \frac{q-1}{2}.$$

Proof. By Lemma 3, we have $\Gamma(H) = \{H^P \mid P \in \mathcal{R}\}$, and it is clear that $H^{P_1} = H^{P_2}$ if and only if $P_2^{-1}P_1 \in H$. Note that

$$P_2^{-1}P_1 = \begin{pmatrix} \alpha_2 & -\dfrac{1}{2\beta_2} \\ \beta_2 & \dfrac{1}{2\alpha_2} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 & -\dfrac{1}{2\beta_1} \\ \beta_1 & \dfrac{1}{2\alpha_1} \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha_2\beta_1 + \alpha_1\beta_2}{2\,\alpha_2\beta_2} & \dfrac{\alpha_2\beta_1 - \alpha_1\beta_2}{4\,\alpha_1\alpha_2\beta_1\beta_2} \\ \alpha_2\beta_1 - \alpha_1\beta_2 & \dfrac{\alpha_2\beta_1 + \alpha_1\beta_2}{2\,\alpha_1\beta_1} \end{pmatrix}.$$

Hence $P_2^{-1}P_1 \in H$ if and only if either $\alpha_1/\alpha_2 = \beta_1/\beta_2$ or $\alpha_1/\alpha_2 = -\beta_1/\beta_2$. Thus,

$$|\Gamma(H)| = |\mathcal{R}/\sim| = \frac{q-1}{2}.$$ □

Lemma 5. *The graph Γ is connected and arc-transitive.*

Proof. By Corollary 2, the graph Γ is vertex-transitive. Moreover, by Statement (4) of Lemma 1, $\mathrm{Aut}(\Gamma)$ has a vertex-primitive subgroup; therefore, Γ is connected.

Thus, it is sufficient to show that $\mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(H)$ acts transitively on $\Gamma(H)$.

Let $H^{P_1} \neq H^{P_2}$, where $P_1, P_2 \in \mathcal{R}$. Since the matrices P_1 and P_2 can be chosen up to equivalence with respect to $\sim$, without loss of generality, we can assume that

$$P_1 = \begin{pmatrix} 1 & -\dfrac{1}{2\beta_1} \\ \beta_1 & \dfrac{1}{2} \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} 1 & -\dfrac{1}{2\beta_2} \\ \beta_2 & \dfrac{1}{2} \end{pmatrix} \text{ for } \beta_1, \beta_2 \in GF(q)^* \text{ with } \beta_1 \neq \beta_2.$$

Let
$$P(\beta_1, \beta_2) = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} \in GL_2(q).$$

Note that $P(\beta_1, \beta_2) \in N_{GL_2(q)}(H)$; therefore, by Proposition 1, conjugation by $P(\beta_1, \beta_2)$ induces a non-trivial automorphism of Γ, which stabilizes H. Moreover, $P_1^{P(\beta_1,\beta_2)} = P_2$; therefore,

$$(H^{P_1})^{P(\beta_1,\beta_2)} = H^{P_2}.$$

Thus, $\mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(H)$ acts transitively on $\Gamma(H)$; therefore, the graph Γ is arc-transitive. □

3. Intersection of two neighborhoods

Lemma 6. *If $Q \in G$, then the number $|\Gamma(H) \cap \Gamma(H^Q)|$ is equal to the number of pairwise non-equivalent with respect to $\sim$ matrices $P \in \mathcal{R}$ such that*

$$P^{-1}QS \in H \text{ for some } S \in \mathcal{R}.$$

Proof. Recall that Γ is vertex-transitive, and, therefore, by Lemma 3, $\Gamma(H) \cap \Gamma(H^Q)$ consists of the matrices H^P, where $P \in \mathcal{R}$, such that there exists a matrix $S \in \mathcal{R}$ with

$$H^P = (H^S)^Q.$$

Now since $N_G(H) = H$ by Statement (1) of Lemma 1, the lemma holds. □

Lemma 7. *If $Q \in G \setminus H$, then $|\Gamma(H) \cap \Gamma(H^Q)| \leq 1$. Moreover, if $Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$, then $|\Gamma(H) \cap \Gamma(H^Q)| = 1$ if and only if $q_{11}q_{21} \neq 0$ and $\dfrac{q_{12}q_{22}}{q_{11}q_{21}}$ is a non-zero square in $GF(q)$.*

Proof. By Lemma 6, $|\Gamma(H) \cap \Gamma(H^Q)|$ is equal to the number of pairwise non-equivalent with respect to $\sim$ matrices $P \in \mathcal{R}$, such that

$$P^{-1}QS \in H$$

for some $S \in \mathcal{R}$. Let

$$P = \begin{pmatrix} x & -\dfrac{1}{2y} \\ y & \dfrac{1}{2x} \end{pmatrix} \in \mathcal{R}, \quad S = \begin{pmatrix} s_1 & -\dfrac{1}{2s_2} \\ s_2 & \dfrac{1}{2s_1} \end{pmatrix} \in \mathcal{R}, \text{ and } Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}.$$

Then

$$M = P^{-1}QS = \begin{pmatrix} \dfrac{q_{21}s_1x + q_{22}s_2x + q_{11}s_1y + q_{12}s_2y}{2\,xy} & -\dfrac{q_{21}s_1x - q_{22}s_2x + q_{11}s_1y - q_{12}s_2y}{4\,s_1s_2xy} \\ q_{21}s_1x + q_{22}s_2x - q_{11}s_1y - q_{12}s_2y & -\dfrac{q_{21}s_1x - q_{22}s_2x - q_{11}s_1y + q_{12}s_2y}{2\,s_1s_2} \end{pmatrix}.$$

Note that $M \in H$ if and only if either $M_{1,1} = M_{2,2} = 0$ or $M_{1,2} = M_{2,1} = 0$.

The equalities $M_{1,1} = M_{2,2} = 0$ imply the equalities

$$q_{21}s_1x + q_{22}s_2x + q_{11}s_1y + q_{12}s_2y = 0 \text{ and } q_{21}s_1x - q_{22}s_2x - q_{11}s_1y + q_{12}s_2y = 0, \tag{3.1}$$

which can be considered as a linear system for $x = x_1$ and $y = y_1$ with the matrix

$$A = \begin{pmatrix} q_{21}s_1 + q_{22}s_2 & q_{11}s_1 + q_{12}s_2 \\ q_{21}s_1 - q_{22}s_2 & -q_{11}s_1 + q_{12}s_2 \end{pmatrix}, \text{ where } \det A = 2(q_{11}q_{21}s_1^2 - q_{12}q_{22}s_2^2).$$

It is clear that there is a non-trivial solution of the system (3.1) if and only if

$$\det A = 0, \text{ i.e., } \frac{s_1^2}{s_2^2} = \frac{q_{12}q_{22}}{q_{11}q_{21}}.$$

Thus, if $\dfrac{q_{12}q_{22}}{q_{11}q_{21}}$ is a non-square in $GF(q)^*$, then the system (3.1) does not have non-zero solutions, and if $\dfrac{q_{12}q_{22}}{q_{11}q_{21}}$ is a square in $GF(q)^*$, then all the matrices $P = P_1$ obtained from the solutions of the system (3.1) are pairwase equivalent with respect to $\sim$. Moreover, since the matrix Q is non-degenerate, we have $q_{11}s_1 + q_{12}s_2 \neq 0$ or $q_{21}s_1 + q_{22}s_2 \neq 0$. Without loss of generality we assume that $q_{21}s_1 + q_{22}s_2 \neq 0$, and then

$$\frac{x_1}{y_1} = -\frac{q_{11}s_1 + q_{12}s_2}{q_{21}s_1 + q_{22}s_2}.$$

The equalities $M_{1,2} = M_{2,1} = 0$ imply the equalities

$$q_{21}s_1x - q_{22}s_2x + q_{11}s_1y - q_{12}s_2y = 0 \text{ and } q_{21}s_1x + q_{22}s_2x - q_{11}s_1y - q_{12}s_2y = 0, \tag{3.2}$$

which can be considered as a linear system for $x = x_2$ and $y = y_2$ with the matrix

$$A = \begin{pmatrix} q_{21}s_1 - q_{22}s_2 & q_{11}s_1 - q_{12}s_2 \\ q_{21}s_1 + q_{22}s_2 & -q_{11}s_1 - q_{12}s_2 \end{pmatrix}, \text{ where } \det A = -2(q_{11}q_{21}s_1^2 - q_{12}q_{22}s_2^2).$$

It is clear that there is a non-trivial solution of the system (3.2) if and only if

$$\det A = 0, \text{ i. e., } \frac{s_1^2}{s_2^2} = \frac{q_{12}q_{22}}{q_{11}q_{21}}.$$

Thus, if $\frac{q_{12}q_{22}}{q_{11}q_{21}}$ is a non-square in $GF(q)^*$, then the system (3.2) does not have non-zero solutions, and if $\frac{q_{12}q_{22}}{q_{11}q_{21}}$ is a square in $GF(q)^*$, then all the matrices $P = P_2$ obtained from the solutions of the system (3.2) are pairwase equivalent with respect to $\sim$. Moreover, again since the matrix Q is non-degenerate, we have $q_{21}s_1 + q_{22}s_2 \neq 0$ or $q_{11}s_1 + q_{12}s_2 \neq 0$. We have assumed before that without loss of generality $q_{21}s_1 + q_{22}s_2 \neq 0$, and then

$$\frac{x_2}{y_2} = \frac{q_{11}s_1 + q_{12}s_2}{q_{21}s_1 + q_{22}s_2}.$$

Now we have proved that if $q_{11}q_{21} = 0$ or $q_{12}q_{22} = 0$ or $\frac{q_{12}q_{22}}{q_{11}q_{21}}$ is a non-square in $GF(q)^*$, then there is no a matrix $P \in \mathcal{R}$ with $PQS^{-1} \in H$ for some $S \in \mathcal{R}$, and if $\frac{q_{12}q_{22}}{q_{11}q_{21}}$ is a square in $GF(q)^*$, then all the matrices $P \in \mathcal{R}$ with $PQS^{-1} \in H$ for some $S \in \mathcal{R}$ are pairwise equivalent with respect to $\sim$. Thus, if $q_{11}q_{21} = 0$ or $q_{12}q_{22} = 0$ or $\frac{q_{12}q_{22}}{q_{11}q_{21}}$ is a non-square in $GF(q)^*$, then $|\Gamma(H) \cap \Gamma(H^Q)| = 0$, and if $\frac{q_{12}q_{22}}{q_{11}q_{21}}$ is a square in $GF(q)^*$, then $|\Gamma(H) \cap \Gamma(H^Q)| = 1$. □

4. Proof of the main results

Note that, by Statement (1) of Lemma 1, $|\mathcal{H}| = |G : N_G(H)| = \frac{q(q+1)}{2}$; thus, Γ has $\frac{q(q+1)}{2}$ vertices. The graph Γ is vertex-transitive by Corollary 2 and is vertex-primitive and connected by Statement (4) of Lemma 1; therefore, it is regular; by Lemma 4, the vertex degree of Γ is $\frac{q-1}{2}$.

By Lemma 5, Γ is arc-transitive; therefore, it is edge-regular. Let $P \in \mathcal{R}$. Then $\frac{p_{12}p_{22}}{p_{11}p_{21}} = \frac{-1}{4\alpha^2\beta^2}$, where α and β are from $GF(q)^*$, which is a square in $GF(q)^*$ by Statement (2) of Lemma 1. Thus, by Lemmas 7 and 3, we conclude that Γ is an amply regular graph with parameters

$$\left(\frac{q(q+1)}{2}, \frac{q-1}{2}, 1, 1\right).$$

Since $Z(G) \leq H$, by Proposition 1 and [2, Table 8.2], we have

$$\mathrm{Aut}(PSL_2(q)) = \mathrm{Aut}(G/Z(G)) = G/Z(G)N_{\mathrm{Aut}(G/Z(G))}(H/Z(G)) \leq \mathrm{Aut}(\Gamma).$$

Let

$$\mathcal{C} = \left\{ H^{Q_a} \middle| Q_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in GF(q) \right\}.$$

Show that $\mathcal{C}$ is a perfect 1-code in Γ. Indeed, $H \in \mathcal{C}$ and by Lemma 3, H is non-adjacent to any other vertex from $\mathcal{C}$. Moreover, by Lemma 7, $|\Gamma(H) \cap \Gamma(A)| = 0$ for each vertex $H \neq A \in \mathcal{C}$. Now note that

$$K = \left\{ Q_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in GF(q) \right\}$$

is a subgroup of G and K acts transitively on $\mathcal{C}$. Thus, for each $A, B \in \mathcal{C}$ with $A \neq B$, we have $|\Gamma(A) \cap \Gamma(B)| = 0$.

Now it is clear that $|\mathcal{C}| = q$ and, since the vertex degree of Γ is $\dfrac{q-1}{2}$ and Γ has exactly $\dfrac{q(q+1)}{2} = q\Big(\dfrac{q-1}{2} + 1\Big)$ vertices, we find that the vertex set of Γ is a disjoint union of balls of radius 1 with centers at the vertices from $\mathcal{C}$. Thus, $\mathcal{C}$ is a perfect 1-code in Γ; therefore, Γ is of diameter more than 2. $\square$

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Mikhail Petrovich Golubyatnikov, Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, Yekaterinburg, 620108 Russia; Ural Federal University, Yekaterinburg, 620000 Russia, e-mail: mike_ru1@mail.ru .

Natalia Vladimirovna Maslova, Dr. Phys.-Math. Sci., Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, Yekaterinburg, 620108 Russia; Prof., Ural Federal University, Yekaterinburg, 620000 Russia, e-mail: butterson@mail.ru .

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