TRUDY INSTITUTA MATEMATIKI I MEKHANIKI UrO RAN

Vol. 28 No. 2 **2022**

MSC: 35Q30, 76D05, 76T10, 76T15

DOI: 10.21538/0134-4889-2022-28-2-143-157

ASSIMILATING DATA ON THE FREE SURFACE OF A FLUID FLOW TO FIND ITS VISCOSITY

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We consider a model of a two-phase immiscible incompressible viscous fluid flow and solve an inverse problem to determine the fluid viscosity from a known location of its free surface. The mathematical model of the fluid flow is reduced to solving a problem described by the Navier–Stokes equation in the field of gravity, the incompressibility equation, and the advection equation for the interface between the two phases and is supplemented by the corresponding initial and boundary conditions. The fluid density and viscosity depend on the spatial coordinates and time. The considered problem is ill-posed, as small errors in the initial data and computations may lead to large errors in the solution. The numerical modeling of such problems requires the use of special methods that guarantee the stability of the computational process with respect to the errors. The aim of this work is to develop methods and algorithms for a stable numerical modeling of the inverse problem. To solve the inverse problem, we propose to use a variational method and to replace the original problem with an extremal problem in which a suitable functional related to the discrepancy between the measurements of the location of the fluid's free surface and its location obtained from the solution of a specially constructed controlled dynamic system is minimized. The desired solution of this extremal problem is successively approximated by solutions of terminal–boundary value control problems for the adjoint system, which represents the gradient of the objective functional. A difficulty of this approach is associated with the numerical simulation of the control problems due to their nonlinearity. Some variants of gradient methods can be applied to minimize the discrepancy functional. The gradient of this functional and the descent step along the anti-gradient are determined analytically, allowing for an essential reduction of computations.

Keywords: viscous fluid, incompressible fluid, two-phase fluid, inverse problem, discrepancy functional, variational method, gradient descent method.

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Received February 2, 2022
Revised March 10, 2022
Accepted March 14, 2022

Funding Agency: This work was supported jointly by the Russian Foundation for Basic Research and the German Research Society (project RFBR no. 20-51-12002, project DFG no. IZ203/14-1).

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Cite this article as: Korotkii A.I., Tsepelev I.A., Ismail-zadeh A.T. Assimilating data on the free surface of a fluid flow to constrain its viscosity. *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2022, vol. 28, no. 2, pp. 143–157.