

MSC: 34E05, 34E10, 34K26, 35K15, 35K59

DOI: 10.21538/0134-4889-2022-28-1-96-110

**MATCHING OF ASYMPTOTIC SOLUTIONS OF A PARABOLIC EQUATION  
IN THE CAUCHY PROBLEM  
WITH THE MULTISCALE EVOLUTION OF A SINGULARITY**

**S. V. Zakharov**

A bisingular Cauchy problem for a quasilinear parabolic equation with a small parameter at the higher derivative is considered. The initial function depends on the space variable with another small parameter, and conditions are fulfilled under which the statement of the problem becomes a model of the evolution of nonlinear compression waves with a large initial gradient in physical systems in the presence of a small dissipation. In the limit case of the zero dissipation, when the equation under consideration becomes a first-order equation, there is a shock wave, whose origin is a singular point of the solution. Near the singular point, on the scales of the small value of dissipation, an asymptotic solution of the problem in the small parameters is constructed. With the help of the matching method on the basis of an earlier obtained asymptotic solution in a smaller region, it is established that the new asymptotic solution must have the form of a series in integer powers of the ratio of small parameters and its logarithm, and for the coefficients of this series a recurrence system of parabolic equations together with the corresponding asymptotic conditions of matching is obtained. After reducing this recurrence system of problems to integral relations, by applying the method of successive approximations and estimates of integral convolutions, the existence of necessary solutions is proved. In addition, it is shown that the constructed series is suitable in a transitional region of the multiscale evolution of the singularity between its initial stage and the boundary layer near the shock wave, and also in the particular case of the Burgers equation some explicit formulas are given.

Keywords: quasilinear parabolic equation, large initial gradient, bisingular Cauchy problem, singular point, multiscale evolution, self-similarity, Poincaré and Erdélyi asymptotics, matching method, dissipative compression wave, shock wave.

**REFERENCES**

1. Poincaré H. Sur les intégrales irrégulières des équations linéaires. *Acta Math.*, 1886, vol. 8, no. 1, pp. 295–344. doi: 10.1007/BF02417092.
2. Erdélyi A. *Asymptotic expansions*. NY: Dover Publ., 1956, 128 p. ISBN: 0486603180.
3. Zakharov S.V. Singular points and asymptotics in the singular Cauchy problem for the parabolic equation with a small parameter. *Comput. Math. Math. Phys.*, 2020, vol. 60, no. 5, pp. 821–832. doi: 10.1134/S0965542520050164.
4. Zakharov S.V. Two-parameter asymptotics in a bisingular Cauchy problem for a parabolic equation. *Proc. Steklov Inst. Math.*, 2018, vol. 301, suppl. 1, pp. S191–S200. doi: 10.1134/S0081543818050164.
5. Zakharov S.V. Cauchy problem for a quasilinear parabolic equation with a large initial gradient and low viscosity. *Comput. Math. Math. Phys.*, 2010, vol. 50, no. 4, pp. 665–672. doi: 10.1134/S0965542510040081.
6. Il'in A.M. *Matching of asymptotic expansions of solutions of boundary value problems*. Providence: AMS, 1992, 281 p. ISBN: 978-0-8218-4561-5. Original Russian text published in *Soglasovanie asimptoticheskikh razlozhenii reshenii kraevykh zadach*. Moscow: Nauka Publ., 1989, 336 p.
7. Whitham G.B. *Linear and nonlinear waves*. NY: John Wiley & Sons Inc., 1974, 636 p. ISBN: 0-471-94090-9.
8. Oleinik O.A. Discontinuous solutions of non-linear differential equations. *Am. Math. Soc., Transl., II*, 1963, vol. 26, pp. 95–172. doi: 10.1090/trans2/026/05.
9. Il'in A.M. On the asymptotics of the solution of a problem with a small parameter. *Math. USSR-Izvestiya*, 1990, vol. 34, no. 2, pp. 261–279. doi: 10.1070/IM1990v034n02ABEH000629.

10. Stewartson K. On almost rigid rotations. Part 2. *J. Fluid Mech.*, 1966, vol. 26, pp. 131–144. doi: 10.1017/S0022112066001137.
11. Busse F.H. On Howard's upper bound for heat transport by turbulent convection. *J. Fluid Mech.*, 1969, vol. 37, pp. 457–477. doi: 10.1017/S0022112069000668.

Received January 18, 2021

Revised October 14, 2021

Accepted October 18, 2021

*Sergei Viktorovich Zakharov*, Cand. Sci. (Phys.-Math.), Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, Yekaterinburg, 620108 Russia, e-mail: svz@imm.uran.ru.

Cite this article as: S.V. Zakharov. Matching of asymptotic solutions of a parabolic equation in the Cauchy problem with the multiscale evolution of a singularity, *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2022, vol. 28, no. 1, pp. 96–110.