TRUDY INSTITUTA MATEMATIKI I MEKHANIKI UrO RAN

Vol. 28 No. 1 **2022**

MSC: 35F21, 35F25

DOI: 10.21538/0134-4889-2022-28-1-257-268

A CONTINUOUS GENERALIZED SOLUTION OF THE HAMILTON–JACOBI EQUATION WITH A THREE-COMPONENT HAMILTONIAN

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The Cauchy problem for the Hamilton–Jacobi equation of evolution type is studied in the case of one-dimensional state space. The domain in which the equation is considered is divided into three subdomains. In each of these subdomains, the Hamiltonian is continuous, and at their boundaries it suffers a discontinuity in the state variable. The Hamiltonian is convex in the impulse variable, and the dependence on this variable is exponential. We define a continuous generalized solution of the Cauchy problem with a discontinuous Hamiltonian on the basis of the viscous/minimax approach. The proof of the existence of such a generalized solution is constructive. First, a viscosity solution is constructed in the closure of the middle domain. Here, the coercivity of the Hamiltonian with respect to the impulse variable in the middle domain is essential. The solution is then continuously extended to the other two domains. The extensions are constructed by solving variational problems with movable ends based on the method of generalized characteristics. The uniqueness of the generalized solution is proved under the condition that the initial function is globally Lipschitz.

Keywords: Hamilton–Jacobi equation, discontinuous Hamiltonian, generalized solutions, viscosity solutions, method of generalized characteristics.

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Received September 16, 2021
Revised January 17, 2022
Accepted January 21, 2022

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Cite this article as: L. G. Shagalova. A continuous generalized solution of the Hamilton–Jacobi equation with a three-component Hamiltonian, *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2022, vol. 28, no. 1, pp. 257–268 .