ON FINITE 4-PRIMARY GROUPS HAVING
A DISCONNECTED GRUENBERG–KEGEL GRAPH
AND A COMPOSITION FACTOR ISOMORPHIC TO $L_3(17)$ or $Sp_4(4)$

A. S. Kondrat'ev, I. D. Suprunenko, I. V. Khramtsov

The Gruenberg–Kegel graph (the prime graph) $\Gamma(G)$ of a finite group G is the graph in which the vertices are the prime divisors of the order of G and two distinct vertices p and q are adjacent if and only if G contains an element of order pq. Investigations of finite groups by the properties of their Gruenberg–Kegel graphs form a dynamically developing branch of the finite group theory. A detailed study of the class of finite groups with disconnected Gruenberg–Kegel graphs is one of the important problems in this direction. In 2010–2011, the first and the third authors described the normal structure of finite 3-primary and 4-primary groups with disconnected Gruenberg–Kegel graphs. Unfortunately, the case where a 4-primary group has a composition factor isomorphic to $L_3(17)$ or $Sp_4(4)$ has been omitted in this description. In the present paper, we obtain a description of the groups under consideration in the omitted case. Now a description of the normal structure of finite 4-primary groups with disconnected Gruenberg–Kegel graphs is corrected. In the course of the proof, the 2-modular decomposition matrix of the group $L_3(17)$ is calculated (up to two parameters every of which takes value 1 or 2).

Keywords: finite group, algebraic group, non-solvable 4-primary group, chief factor, disconnected Gruenberg–Kegel graph, character, Brauer character, decomposition matrix.

REFERENCES

1. Belonogov V.A. *Predstavleniya i kharaktery v teorii konechnykh grupp* [Representations and characters in the theory of finite groups]. Sverdlovsk: UrO AN SSSR Publ., 1990, 380 p. ISBN: 9785769100758 .

2. Bourbaki N. *Groupes et algèbres de Lie* (*Chapt. IV–VI*). Paris: Hermann, 1968, 282 p. doi: 10.1007/978-3-540-34491-9 . Translated into Russian under the title *Gruppy i algebry Li* (*glavy IV–VI*, Moscow: Mir Publ., 1972, 334 p.

3. Bourbaki N. *Groupes et algèbres de Lie, Chaps. VII–VIII*. Berlin; Heidelberg: Springer-Verlag, 2006, 265 p. doi: 10.1007/978-3-540-33977-9 . Translated into Russian under the title *Gruppy i algebry Li, glavy VII–VIII*, Moscow: Mir Publ., 1978, 342 p.

4. Kondrat'ev A.S. Prime graph components of finite simple groups. *Math. USSR Sb.*, 1990, vol. 67, no. 1, pp. 235–247. doi: 10.1070/SM1990v06n01ABEN001363 .

5. Kondrat'ev A.S., Khramtsov I.V. On finite triprimary groups. *Trudy Inst. Mat. i Mekh. UrO RAN*, 2010, vol. 16, no. 3, pp. 150–158 (in Russian).

6. Kondrat'ev A.S., Khramtsov I.V. On finite tetraprimary groups. *Proc. Steklov Inst. Math.*, 2012, vol. 279, no. 1, pp. 43–61. doi: 10.1134/S0081543812090040 .

7. Curtis C.W., Reiner I. *Representation theory of finite groups and associative algebras.* American Mathematical Society, 2006, 689 p. ISBN: 978-0821840665 . Translated into Russian under the title *Teoriya predstavlenii konechnykh grupp i assotsiativnykh algebr*, Moscow: Nauka Publ., 1969, 668 p.

8. Steinberg R. *Lectures on Chevalley groups.* American Mathematical Society, 2016, 160 p. ISBN: 978-1470431051 . Translated into Russian under the title *Lektsii o gruppakh Shevalle*, Moscow: Mir Publ., 1975, 262 p.

9. Suprunenko I.D. The invariance of the weight systems of algebraic groups and Lie algebras of type A_l with restricted highest weights under reduction modulo p. *Izv. Akad. Nauk BSSR, Ser. Fiz.-Mat. Nauk*, 1983, no. 2, pp. 18–22 (in Russian).

10. Arad Z., Herfort W. Classification of finite groups with a CC-subgroup. *Comm. Algebra*, 2004, vol. 32, no. 6, pp. 2087–2098. doi: 10.1081/AGB-120037209 .

11. Conway J.H. et al. *Atlas of finite groups*. Oxford: Clarendon Press, 1985, 252 p. ISBN: 0198531990 .

12. Jansen C. et al. *An atlas of Brauer characters*. Oxford : Clarendon Press, 1995, 327 p. ISBN: 0198514816 .

13. Baranov A.A., Osinovskaya A.A., Suprunenko I.D. Modular representations of the special linear groups with small weight multiplicities. *J. Algebra*, 2014, vol. 397, pp. 225–251. doi: 10.1016/j.jalgebra.2013.08.032 .

14. Bray J.N., Holt D.F., Roney-Dougal C.M. *The maximal subgroups of the low-dimensional finite classical groups*. London Math. Soc. Lect. Note Ser., vol. 407. Cambridge: Cambridge University Press, 2013, 438 p. doi: 10.1017/CBO9781139192576 .

15. Craven D.A. *Representation theory of finite groups: a guidebook*. Universitext. Cham: Springer, 2019. 294 p. ISBN: 98-3-030-21791-4.

16. Dipper R. On the decomposition numbers of the finite general linear groups. II. *Trans. Amer. Math. Soc.*, 1985, vol. 292, no. 1, pp. 123–133. doi: 10.1090/S0002-9947-1985-0805956-7 .

17. Dornhoff L.L. *Group representation theory. Part B: Modular representation theory*. NY: M. Dekker, 1971, 256 p. ISBN: 0824711483 .

18. The GAP Group, *GAP − Groups, Algorithms, and Programming, Ver. 4.11.1. 2021.* Available on: http://www.gap-system.org .

19. Huppert B. *Endliche Gruppen I.* Berlin etc.: Springer, 1967, 793 p. doi: 10.1007/978-3-642-64981-3 .

20. Huppert B., Blackburn N. *Finite groups II.* Berlin; NY: Springer, 1982, 531 p. ISBN: 0387106324 .

21. Iiyori N., Yamaki H. Prime graph components of the simple groups of Lie type over the fields of even characteristic. *J. Algebra*, 1993, vol. 155, no. 2, pp. 335–343; Corrigenda. *J. Algebra*, 1996, vol. 181, no. 2, p. 659. doi: 10.1006/jabr.1996.0140 .

22. Jantzen J.C. *Representations of algebraic groups: second edition.* Providence, R.I.: Amer. Math. Soc., 2003, 576 p. ISBN: 978-0-8218-4377-2 .

23. Lucido M.S. Prime graph components of finite almost simple groups. *Rend. Sem. Mat. Univ. Padova*, 1999, vol. 102, pp. 1–22; Addendum. *Rend. Sem. Mat. Univ. Padova*, 2002, vol. 107, pp. 189–190.

24. Mortimer B. The modular permutation representations of the known doubly transitive groups. *Proc. London Math. Soc.*, 1980, vol. 41, no. 1, pp. 1–20. doi: 10.1112/plms/s3-41.1.1 .

25. Simpson W., Frame J.S. The character tables for $SL(3, q)$, $SU(3, q^2)$, $PSL(3, q)$, $PSU(3, q^2)$. *Canad. J. Math.*, 1973, vol. 25, no. 3, pp. 486–494. doi: 10.4153/CJM-1973-049-7 .

26. Steinberg R. Representations of algebraic groups. *Nagoya Math. J.*, 1963, vol. 22, pp. 33–56. doi: 10.1017/S0027763000011016 .

27. Williams J. Prime graph components of finite groups. *J. Algebra*, 1981, vol. 69, no. 2, pp. 487–513. doi: 10.1016/0021-8693(81)90218-0 .

Anatolii Semenovich Kondrat'ev, Dr. Phys.-Math. Sci., Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, Yekaterinburg, 620108 Russia, e-mail: A.S.Kondratiev@imm.uran.ru .

Irina Dmitrievna Suprunenko, Dr. Phys.-Math. Sci., Institute of Mathematics of the National Academy of Sciences of Belarus, Minsk, 200072 Belarus, e-mail: suprunenko@im.bas-net.by .

Igor Vladimirovich Khramtsov, Cand. Sci. (Phys.-Math.), OOO "Yandex Tehnology", Moscow, 119021 Russia, e-mail: ihramtsov@gmail.com .