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ON BORWEIN'S IDENTITY AND WEIGHTED TURÁN TYPE INEQUALITIES ON A CLOSED INTERVAL

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Let Π_n^* be the class of algebraic polynomials P of degree n having all zeros on the interval $[-1,1]$ and vanishing at the points 1 and -1. In addition, let $w(x) = 1 - x^2$. The main result of the paper can be formulated as follows: there is an absolute constant $A > 0$ such that

$$\|P' w^{1-s}\|_{C[-1,1]} > A\sqrt{n} \cdot \sqrt{1 - \Delta_P^2}\, \|P w^{-s}\|_{C[-1,1]}$$

for any $P \in \Pi_n^*$ and $s \in [0,1]$, where $\Delta_P = \inf\{d \geq 0 \colon \|P w^{-s}\|_{C[-d,d]} = \|P w^{-s}\|_{C[-1,1]}\}$. This inequality may be interpreted as a weighted analog of P. Turán's classical inequality for the derivative of polynomials with zeros on a closed interval. The proof uses a generalization of an interesting formula of P. Borwein concerning the logarithmic derivative of such polynomials. Our estimate is sharp in the order of the quantity n and complements well-known results of V. F. Babenko, S. A. Pichugov, S. P. Zhou, and others.

Keywords: logarithmic derivative of a polynomial, weighted Turán inequality.

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