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ENUMERATION TECHNIQUES ON CYCLIC SCHUR RINGS¹

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Any Schur ring is uniquely determined by a partition of the elements of the group. In this paper we present a general technique for enumerating Schur rings over cyclic groups using traditional Schur rings. We also survey recent efforts to enumerate Schur rings over cyclic groups of specific orders.

Keywords: Schur ring, cyclic group, association scheme, lattice of subgroups.

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