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ON THE BEST SIMULTANEOUS POLYNOMIAL APPROXIMATION OF FUNCTIONS AND THEIR DERIVATIVES IN HARDY SPACES

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In this paper, we solve extremal problems related to the best simultaneous polynomial approximation of functions analytic in the unit disk and belonging to the Hardy space $\mathscr{H}_2$. The problem of simultaneous approximation of periodic functions by trigonometric polynomials was considered by A. L. Garkavi in 1960. Then, in the same year, A. F. Timan considered this problem for classes of entire functions defined on the axis. We establish a number of exact theorems and calculate the exact values of the least upper bounds of the best simultaneous approximations of a function and its successive derivatives by polynomials and their corresponding derivatives on some classes of complex functions belonging to the Hardy space $\mathscr{H}_2$.

Keywords: best simultaneous approximation, analytic function, unit disk, modulus of continuity, extremal problem, angular boundary value, polynomial.

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