Trudy Instituta Matematiki i Mekhaniki UrO RAN

Vol. 27 No. 3 2021

WILCOX FORMULA FOR VECTOR FIELDS ON BANACH MANIFOLDS

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We obtain an analogue of Wilcox–Snyder formula for flows of diffeomorphisms of C^m-smooth vector fields on infinite-dimensional Banach manifolds. For classical linear system this formula can be efficiently used, for example, to obtain Magnus expansion of solutions. The generalized Wilcox formula is obtained by using an extended Chronological Calculus for Banach manifold. We apply this formula to derive new structured differential equations which solutions approximate solutions of the original differential equation.

Keywords: flow of diffeomorphisms, Wilcox formula, chronological calculus, Magnus expansion.

Ю. С. Ледяев. Формула Уилкокса для векторных полей на банаховых многообразиях.

Получен аналог формулы Уилкокса — Снайдера для потоков диффеоморфизмов C^m-гладких векторных полей на бесконечномерных банаховых многообразиях. Эта формула может эффективно использоваться, например, для получения разложения Магнуса решений классических линейных систем. Обобщенная формула Уилкокса получена с использованием расширения хронологического исчисления для банаховых многообразий. Эта формула применена для вывода новых стуктурированных дифференциальных уравнений, решения которых приближают решения исходного дифференциального уравнения.

Ключевые слова: поток диффеоморфизмов, формула Уилкокса, хронологическое исчисление, разложение Магнуса.

MSC: 93C10, 45G15

DOI: 10.21538/0134-4889-2021-27-3-271-285

1. Introduction

We consider a flow of diffeomorphisms on C^m-smooth Banach manifolds M which is defined by the following differential equation:

$$\frac{d}{dt}q(t) = V_t(q(t)), \quad q(t) \in M, \tag{1.1}$$

where $V_t(q)$ is a family of C^m-smooth vector fields on the manifold M for almost all t which are measurable in t for all $q \in M$.

For such flows we derive a generalization of Wilcox–Snider formula from [14; 15] for nonlinear dynamical system (1.1). Wilcox formula can be used in many applications, in particular, in [15] Wilcox suggested an approach to a derivation of Magnus expansions [12] of solutions of linear differential operator equations. Such Magnus expansions have many applications (see [1; 2; 5; 6]).

From this point of view the present paper's objective is to derive Wilcox formula for nonlinear dynamical systems (1.1) on Banach manifolds which can serve as a technical foundation for obtaining approximations in more general setting under less restrictive assumptions than it was done before for nonlinear system (1.1).

It is important to mention that the Magnus expansion and the original Wilcox formula were obtained for linear systems.

Here we use methods of Chronological Calculus [1; 2] by Agrachev and Gamkrelidze to rewrite the differential equation (1.1) in the form of linear differential operator equation. The original Chronological Calculus was developed for C^∞-smooth finite-dimensional manifolds and vector fields.

This paper relies on the extension [10] of Chronological Calculus for C^m-smooth Banach manifolds and vector fields which are only measurable in time.

The paper consists of four sections. The second sections contains a brief description of classical results on linear systems, Magnus expansions and Wilcox formula. We put them here in order to make the content of this note more accessible to a wider readers' audience. We also put a short proof of the original Wilcox formula by using elementary tools of differential equations theory. This proof will serve as an example for the proof of a generalization of Wilcox formula for nonlinear systems in Section 4.

Section 3 contains a brief description of extension of Chronological Calculus from [10] and appropriate techniques which will be used in a derivation of the generalization of Wilcox formula for the system (1.1). Section 4 contains the proof of this generalization and the last Section contains a construction of approximations of solutions of (1.1) in terms of solutions of some structured dynamical systems for small and large times. These structured differential equations are written in terms of iterated Lie brackets of the original vector field in (1.1). We can see some possible applications of these results to problems of singular control problems, to problems of controllability and stabilizability of nonlinear systems,

2. Linear Systems, Magnus Expansions and Wilcox Formula

Let $\mathbb{X}$ be a Banach space, $A : \mathbb{X} \to \mathbb{X}$ be a bounded linear operator. We consider a linear evolution equation [9] (or differential linear operator equation)

$$\dot{x} = Ax, \quad x(0) = x_0. \tag{2.1}$$

Under appropriate assumptions on A the solution of the initial-value problem (2.1) can be written as

$$x(t) = e^{tA}x_0,$$

where operator (matrix) exponential is defined by the following infinite series ($I : \mathbb{X} \to \mathbb{X}$ is the identity operator):

$$e^{tA} = I + tA + \frac{t^2A^2}{2!} + \ldots + \frac{t^kA^k}{k!} + \ldots .$$

Note that e^{tA} is a fundamental operator (matrix) solution of (2.1)

$$\left(e^{tA}\right)' = Ae^{tA}. \tag{2.2}$$

When $\mathbb{X} = \mathbb{R}^n$ the operator A becomes $n \times n$ matrix.

In the case of nonautonomous linear evolution equation the solution of Cauchy problem

$$\dot{x}(t) = A(t)\, x(t), \quad x(0) = x_0,$$

can be written in the form

$$x(t) = X(t)x_0,$$

where the fundamental operator (matrix) solution of (2.2) can be written as Volterra series

$$X(t) = I + \int_0^t A(t_1)\, dt_1 + \int_0^t A(t_1)\Big(\int_0^{t_1} A(t_2)\, dt_2\Big)\, dt_1 + \ldots . \tag{2.3}$$

Note that $X(t)$ in (2.3) satisfies the linear evolution equation

$$X(t)' = A(t)X(t), \quad X(0) = I. \tag{2.4}$$

We remind that in quantum physics series similar to (2.5) are called Dyson series [8].

Volterra expansion (2.3) of the fundamental solution $X(t)$ of the linear evolution equation (2.4) is not very convenient for approximation of the solution by using partial sums in (2.3) since in many cases these partial sums don't satisfy conservation laws arising from symmetries of the system.

But there is another representation of the fundamental solution of (2.4) which lacks such disadvantage. This representation which was introduced by Magnus in [12] is called *Magnus expansion* and has form

$$X(t) = e^{\Omega(t)}, \tag{2.5}$$

where $\Omega(t)$ is an infinite series

$$\Omega(t) := \Omega_1(t) + \Omega_2(t) + \ldots + \Omega_k(t) + \ldots \tag{2.6}$$

and first three terms are

$$\begin{aligned}
\Omega_1(t) &:= \int_0^t A(t_1)\, dt_1, \\
\Omega_2(t) &:= \frac{1}{2}\int_0^t dt_1 \int_0^{t_1} dt_2 [A(t_1), A(t_2)], \\
\Omega_3(t) &:= \frac{1}{3!}\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \Big(\big[A(t_1),[A(t_2),A(t_3)]\big] + \big[A(t_3),[A(t_2),A(t_1)]\big]\Big).
\end{aligned} \tag{2.7}$$

We use classical notation $[A, B]$ to denote the commutator of two operators A and B

$$[A, B] := AB - BA.$$

We also remind the linear operator $\mathrm{ad}(A)$ acting on the space of linear operators

$$\mathrm{ad}(A)X := [A, X].$$

Note that $(\mathrm{ad}(A))^0 X := X,\ (\mathrm{ad}(A))^k X := [A, (\mathrm{ad}(A))^{k-1} X],\ k = 1, 2, \ldots$. It is easy to check that for any k-th derivative $\big(e^{tA} X e^{-tA}\big)^{(k)} = e^{tA}(\mathrm{ad}(A))^k X e^{-tA}$. It follows immediately that

$$e^{tA} X e^{-tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\mathrm{ad}(A))^k X = e^{t\mathrm{ad}(A)} X. \tag{2.8}$$

In many physical applications by replacing $\Omega(t)$ in (2.6) by the partial sum we obtain the approximation of Magnus expansion of fundamental solution (2.5) which maintains its important physical properties. For example in the case $\mathbb{X} = \mathbb{R}^n$, if matrices $A(t)$ are skew-symmetric then values of approximations of Magnus expansion (where $\Omega(t)$ is replaced by partial sums of (2.6)) are orthogonal matrices.

This remarkable fact will be explained later in the context of dynamical systems (1.1) on manifolds which describe evolution of physical systems subject to some conservation laws.

An excellent exposition of theory of Magnus expansions and its applications can be found in a survey [4] which also contains a detailed bibliography.

Here we provide a brief explanation of a derivation of Magnus expansion formula (2.5)–(2.6) to illustrate the use of Wilcox formula.

The standard approach is based on the following expression for the derivative of the function $A \to e^A$:

$$de^A = e^A\, \frac{I - e^{-\mathrm{ad}(A)}}{\mathrm{ad}(A)}. \tag{2.9}$$

Rossmann [13] relates original versions of this result to F.Shur and Poincaré.

Note that the operator-valued function $\dfrac{I-e^{-\mathrm{ad}(A)}}{\mathrm{ad}(A)}$ is determined by the power series for the function

$$\frac{1-e^{-x}}{x}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}x^k,$$

where x is replaced by the linear operator $\mathrm{ad}(A)$.

Then for differentiable operator-valued function $A(t)$ we easily obtain from (2.9) that

$$\left(e^{A(t)}\right)'=e^{A(t)}\ \frac{I-e^{-\mathrm{ad}(A(t))}}{\mathrm{ad}(A(t))}A'(t).$$

From this formula we obtain by using product rule for derivative of the next product $\left(e^{A(t)}e^{-A(t)}\right)'=0$ that

$$\left(e^{A(t)}\right)'=\frac{e^{\mathrm{ad}(A(t))}-I}{\mathrm{ad}(A(t))}A'(t)e^{A(t)}. \tag{2.10}$$

Returning to Magnus expansion (2.5), we obtain from (2.10) and (2.4) that $\Omega'(t)=\dfrac{\mathrm{ad}(\Omega(t))}{e^{\mathrm{ad}(\Omega(t))}-I}A(t)$, where we have the following representation with Bernoulli numbers B_k:

$$\frac{\mathrm{ad}(\Omega(t))}{e^{\mathrm{ad}(\Omega(t))}-I}=\sum_{k=0}^{\infty}\frac{B_k}{k!}\big(\mathrm{ad}(\Omega(t))\big)^k.$$

By using previous relations we obtain the following equation [4]: $\Omega'(t)=\displaystyle\sum_{k=0}^{\infty}\frac{B_k}{k!}\big(\mathrm{ad}(\Omega(t))\big)^kA(t)$ which can be used to derive recursively expressions for $\Omega_k(t)$ in the Magnus expansion (2.6)–(2.7).

In his 1967 paper [15] Wilcox suggested a different approach to the derivation of Magnus expansion based on the following Wilcox formula (2.11).

Let us consider an operator-valued function $H(\alpha)$ of a scalar variable α, $H(\alpha):\mathbb{X}\to\mathbb{X}$ is a linear bounded operator for any α. We assume that $H(\alpha)$ is differentiable at α and

$$H'(\alpha)=\lim_{\Delta\alpha\to 0}\frac{H(\alpha+\Delta\alpha)-H(\alpha)}{\Delta\alpha}.$$

Theorem 2.1 (Wilcox formula). *Under previous assumption the following derivative exists:*

$$\frac{\partial}{\partial\alpha}e^{tH(\alpha)}=\int_0^t e^{(t-s)H(\alpha)}H'(\alpha)e^{sH(\alpha)}\,ds. \tag{2.11}$$

Proof. Let $Y_{\Delta\alpha}(t):=e^{tH(\alpha+\Delta\alpha)}$ be a fundamental solution of the linear evolution equation $Y'_{\Delta\alpha}(t)=H(\alpha+\Delta\alpha)Y_{\Delta\alpha}(t), Y_{\Delta\alpha}(0)=I$, and $Y_0(t):=e^{tH(\alpha)}$ be a solution to $Y_0'(t)=H(\alpha)Y_0(t)$, $Y_0(0)=I$. It is easy to see that

$$Y'_{\Delta\alpha}(t)=H(\alpha)Y_{\Delta\alpha}(t)+\Delta H(\Delta\alpha)Y_{\Delta\alpha}(t),\quad \Delta H(\Delta\alpha):=H(\alpha+\Delta\alpha)-H(\alpha).$$

By using Cauchy formula for the previous linear evolution equation we obtain that

$$Y_{\Delta\alpha}(t)=Y_0(t)+\int_0^t e^{(t-s)H(\alpha)}\Delta H(\Delta\alpha)Y_{\Delta\alpha}(s)\,ds.$$

This implies that for any $\Delta\alpha \neq 0$

$$\frac{Y_{\Delta\alpha}(t) - Y_0(t)}{\Delta\alpha} = \int\limits_0^t e^{(t-s)H(\alpha)} \frac{\Delta H(\Delta\alpha)}{\Delta\alpha} Y_{\Delta\alpha}(s)\ ds.$$

By taking limit as $\Delta\alpha \to 0$, we obtain (2.11). □

Note that the proof of the Wilcox formula is based on an idea of differentiation of solution of differential equation by parameter. In this case the differential equation is linear. But the same idea can be used for a derivation of a generalized Wilcox formula for the nonlinear equation (1.1).

Now we demonstrate how to use Wilcox formula (2.11) to derive the formula (2.9) for the derivative of operator exponential function.

Let us fix some bounded linear operator $Y : \mathbb{X} \to \mathbb{X}$ and define operator

$$H(\alpha) := A + \alpha Y$$

then $H(0) = A, H'(0) = Y$ and we obtain from the Wilcox formula (2.11) (with $t = 1$) the expression for Gateau derivative of the operator exponential (by using (2.8))

$$\Big(\frac{\partial}{\partial\alpha} e^{H(\alpha)}\Big)_{\alpha=0} = \int\limits_0^1 e^{(1-s)A} Y e^{sA}\ ds = e^A \int\limits_0^1 e^{-s\mathrm{ad}(A)} Y\ ds$$

$$= e^A \Big(\int\limits_0^1 \sum_{k=0}^{\infty} \frac{s^k}{k!} (-\mathrm{ad}(A))^k\ ds \Big) Y = e^A \sum_{k=0}^{\infty} \frac{s^{k+1}}{(k+1)!} (-\mathrm{ad}(A))^k Y = e^A \frac{I - e^{-\mathrm{ad}(A)}}{\mathrm{ad}(A)} Y,$$

which implies (2.9) since Y is arbitrary.

3. Extended Chronological Calculus

Here we collect notation and some results of Chronological Calculus which will be used in this paper.

Let E and F be Banach spaces. Recall that a map $f : E \to F$ is said to be *differentiable* at x_0 if there exists a bounded linear operator $f'(x_0) : E \to F$ such that for all $x \in E$ we have $f(x) = f(x_0) + f'(x_0)(x - x_0) + o(\|x - x_0\|)$. If f is differentiable on all of E, then we have $f' : E \to L(E, F)$, where $L(E, F)$ is the Banach space of bounded linear operators from E to F. When f' is continuous, we say that f is of class C^1. As a map between Banach spaces, we may then ask if f' is differentiable and so on. If f has m continuous derivatives, then we say that f is of class C^m.

Functions $f : \mathbb{R} \to E$ which take values in a Banach space can also be integrated. For a rigorous introduction to the integration of vector-valued functions, we recommend [7]. This monograph contains detailed description of *Bochner* integrable functions and *Bochner* integral. A brief synopsis can be also found in [10].

It is worth noting that when $E = \mathbb{R}^n$, the Bochner integral is the same as the Lebesgue integral. In general Banach spaces, the Bochner integral retains many desirable properties of the Lebesgue integral. In particular, one has

$$\frac{d}{dt} \int\limits_{t_0}^t f(\tau), \quad d\tau = f(t),$$

for almost all t in $[t_0, t_1]$. Function $F(t)$ is called *absolutely continuous* if $F(t) = F(t_0) + \int_{t_0}^t f(\tau)\, d\tau$ for some integrable f.

We remind some results from the theory of differential equations in Banach spaces

$$\dot{x} = f(t,x), \quad x(t_0) = x_0, \tag{3.1}$$

where $f : J \times E \to E$ and $J \subseteq \mathbb{R}$ is an interval containing t_0. An excellent resource for this material is [6]. There it is demonstrated that in a Banach space, continuity of f is not enough to ensure a solution. We introduce the following definitions for vector fields on E:

Definition 3.1. A *nonautonomous C^m vector field* on E is a function $f : J \times E \to E$ which is measurable in t for each fixed x and C^m in x for almost all t.

Definition 3.2. A nonautonomous C^m vector field on E is said to be *locally integrable bounded* if for any $x_0 \in E$, there exists an open neighborhood U of x_0 and $k \in L^1(J,\mathbb{R})$ such that for all $x \in U$, for all $0 \le i \le m$, we have $\|f^{(i)}(t,x)\| \le k(t)$ for almost all t, where $f^{(i)}$ denotes the i^{th} derivative of f with respect to x.

It can be shown that if $f : J \times E \to E$ is a nonautonomous C^m vector field that is locally integrable bounded, then for any (t_0, x_0) there exists an open interval $J_0 \subset J$ containing t_0 and depending on (t_0, x_0) as well as a unique, absolutely continuous function $x : J_0 \to E$ which satisfies (3.1) for almost all $t \in J_0$. This type of solution is called a *Carathéodory* solution. In addition, the dependence of this solution upon the initial condition x_0 is C^m-smooth. More precisely, if $x(t; t_0, x_0)$ denotes the solution to (3.1), then $x_0 \mapsto x(t; t_0, x_0)$ is m times continuously differentiable for appropriate values of t and x_0.

We will write $P_{t_0,t}$ for the *local flow* $x_0 \mapsto x(t; t_0, x_0)$. Uniqueness of solutions gives us the following properties for the flow:

$$\begin{aligned} P_{s,t} \circ P_{t_0,s}(x) &= P_{t_0,t}(x), \\ P_{t_0,t}^{-1}(x) &= P_{t,t_0}(x). \end{aligned} \tag{3.2}$$

In defining dynamical systems, it is enough for the underlying space to have the structure of a Banach space only locally. Here we remind the reader of some definitions and basic results from the theory of smooth manifolds. For a greater level of detail, we suggest [11].

A *Banach manifold* of class C^m over a Banach space E is a paracompact Hausdorff space M along with a collection of coordinate charts $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$, where A is an indexing set. This collection of charts should be such that the collection $\{U_\alpha\}$ is a cover for M; each φ_α is a bijection of U_α with an open subset of E; and the transition maps $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are of class C^m.

If M and N are Banach manifolds, a function $f : M \to N$ is said to be *C^m-smooth* (or C^m for brevity) if for any coordinate charts $\varphi : U \subseteq M \to E$ and $\psi : V \subseteq N \to F$ the map $\psi \circ f \circ \varphi^{-1}$ is a C^m-smooth mapping of Banach spaces. Analogously, a function $f : M \to N$ is *differentiable* at a point q_0 if $\psi \circ f \circ \varphi^{-1}$ is differentiable at $\varphi(q_0)$.

The *tangent space* to M at q is defined as follows. Consider the collection of differentiable curves $\gamma : \mathbb{R} \to M$ with $\gamma(0) = q$ and define an equivalence relation on this collection by $\gamma_1 \sim \gamma_2$ if and only if $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$ for some coordinate chart φ. One can check that if this relationship holds for one coordinate chart, it will hold for all coordinate charts. We write $[\gamma]$ for the equivalence class of a curve γ. The collection of these equivalence classes forms the tangent space T_qM and there is a natural isomorphism $T_qM \leftrightarrow E$.

Every C^m map $f : M \to N$ induces a map from T_qM to $T_{f(q)}N$ by $[\gamma] \mapsto [f \circ \gamma]$ and we denote this mapping by $f_*(q)$. The tangent bundle TM is a union of the tangent spaces with a topology given locally by the charts $(q, v) \mapsto (\varphi(q), \varphi_*(q)v)$, where φ is a coordinate chart for M. When f is a map between linear spaces E and F, we will write f' for its derivative. When f is a map between Banach manifolds, we will write f_* for the corresponding map from TM to TN. We emphasize that in local coordinates, $f_*(q) : T_qM \to T_{f(q)}N$ is the map given by $v \mapsto f'(q)v$. In contrast, the map $f_* : TM \to TN$ sends a pair (q, v) to the pair $(f(q), f_*(q)v)$.

Let $\pi : TM \to M$ be the projection $(q, v) \mapsto q$. A *nonautonomous vector field* is a mapping $V : \mathbb{R} \times M \to TM$ which satisfies $\pi \circ V_t(q) = q$. Given $q_0 \in M$ and a coordinate chart (φ, U) at q_0, the function $J \times E \to E$ given by

$$(\varphi_* V_t)(x) := \varphi_* \left(\varphi^{-1}(x)\right) V_t \left(\varphi^{-1}(x)\right) \tag{3.3}$$

is the *local coordinate representation for* V_t. Recalling definition 3.2 we introduce

Definition 3.3. A nonautonomous vector field on M is said to be a *locally integrable bounded* C^k *vector field* if it is C^k-smooth in q for almost all t, is measurable in t, and in some neighborhood of each $q \in M$ there is a coordinate representation (3.3) which is locally integrable bounded.

If $x(t)$ is a solution for the differential equation $\dot{x} = (\varphi_* V_t)(x)$ on E with initial condition $x(t_0) = \varphi(q_0)$, then $q(t) = \varphi^{-1} \circ x(t)$ is a solution to the differential equation on M

$$\dot{q} = V_t(q), \quad q(t_0) = q_0. \tag{3.4}$$

For any $\varphi \in C^m(M, E)$ we have the following integral representation:

$$\varphi(q(t)) = \varphi(q_0) + \int_{t_0}^{t} \varphi_*(q(\tau)) V_\tau(q(\tau))\, d\tau. \tag{3.5}$$

With each nonautonomous vector field V_t on M, we associate a local flow $P_{t_0,t}$ given by $q_0 \mapsto q(t; t_0, q_0)$, the solution to (3.4) with initial condition $q(t_0) = q_0$. These flows are C^m diffeomorphisms of M and are of central importance in the development of our extension of the Chronological Calculus, which we now turn to.

The main observation behind the Chronological Calculus [1–3] is that one may trade analytic objects such as diffeomorphisms and vector fields for algebraic objects such as automorphisms and derivations of the algebra $C^\infty(M)$, which is the collection of C^∞ mappings $f : M \to \mathbb{R}$. This correspondence is developed in [1–3], where $C^\infty(M)$ is given the structure of a Fréchet space.

In [10] we developed a streamlined version of the theory which is effective for computations with infinite-dimensional C^m-manifolds and dynamical systems. In order to include Banach spaces in the theory, we consider the vector space $C^m(M, E)$ of C^m functions $f : M \to E$ rather than the algebra of scalar functions $C^\infty(M)$.

We begin by defining the following operators:

i. The *identity* operator $\widehat{Id}$ is defined as follows $\widehat{Id}(\varphi) = \varphi$ for any $\varphi \in C(M, E)$.

ii. Given any point $q \in M$, let $\widehat{q} : C^m(M, E) \to E$ be the linear map given by $\widehat{q}(\varphi) := \varphi(q)$.

iii. Given C^m-manifolds M and N over a Banach space E and a C^m map $P : M \to N$, let $\widehat{P} : C^m(N, E) \to C^r(M, E)$ $(0 \le r \le m)$ be the linear map given by $\widehat{P}(\varphi) := \varphi \circ P$. Note that if P is a diffeomorphism of M, $\widehat{P}$ gives us an isomorphism of $C^m(M, E)$.

iv. Given a tangent vector $v \in T_q M$, let $\widehat{v} : C^m(M, E) \to E$ be the linear map given by $\widehat{v}(\varphi) := \varphi_*(q)v$.

v. Given any C^m vector field V on M, we define a linear map $\widehat{V} : C^m(M, E) \to C^{m-1}(M, E)$ by $\widehat{V}(\varphi) : q \mapsto \varphi_*(q)V(q)$.

Of course, we can consider linear combinations of such linear operators.

When φ is a local diffeomorphism, these operators simply give local coordinate expressions. We need not restrict ourselves to the space $C^m(M, E)$. Indeed, given any open set $U \subseteq M$, we may view U as a Banach manifold in its own right and therefore consider the space $C^m(U, E)$. For example, the local flow $P_{t_0,t} : J_0 \times U \to U$ of a vector field V_t gives rise to a family of linear mappings $\widehat{P}_{t_0,t} : C^m(U, E) \to C^m(U, E)$.

Note that for operators $\widehat{P}$ the semigroup property (3.2) for flow of diffeomorphism $P_{t_0,t}$ becomes

$$\widehat{P}_{t_0,s} \circ \widehat{P}_{s,t} = \widehat{P}_{t_0,t}.$$

Consider an operator-valued function $t \to A_t$ whose values are linear mappings $A_t : C^m(M,E) \to C^p(M,E)$. This function is called *integrable* if for any $\varphi \in C^m(M,E)$ and $q \in M$ the function $t \to A_t(\varphi)(q)$ is integrable. Then the linear operator $\left(\int_{t_0}^{t_1} A_\tau \, d\tau\right) : C^m(M,E) \to C^r(M,E)$ is defined as follows

$$\left(\int\limits_{t_0}^{t_1} A_\tau \, d\tau\right)(\varphi)(q) := \int\limits_{t_0}^{t_1} A_\tau(\varphi)(q)\, d\tau.$$

It follows immediately from (3.4) and (3.5) that the flow operator $\widehat{P}_{t_0,t}$ representing flow of diffeomorphisms for a nonautonomous vector field V_t satisfies the integral equation

$$\widehat{P}_{t_0,t} = \widehat{Id} + \int\limits_{t_0}^{t} \widehat{P}_{t_0,\tau} \circ \widehat{V}_\tau \, d\tau. \tag{3.6}$$

Moreover, we have that the unique operator valued solution of the integral equation (3.6) is the function $t \to \widehat{P}_{t_0,t}$.

In the case when the vector field V_t is only integrable in t then a Carathéodory solution $q(t)$ of the differential equation (3.4) is an absolutely continuous function and we cannot guarantee that $\widehat{P}_{t_0,t}$ is differentiable for every t.

An operator-valued function $\widehat{A}_t$ is called *absolutely continuous* on $[a,b]$ if $\widehat{A}_t = \widehat{A}_{t_0} + \int_{t_0}^{t} \widehat{B}_\tau \, d\tau$ for any $t \in [a,b]$ for some integrable operator-valued function $\widehat{B}_t$. We denote $\widehat{B}_t$ as $\dfrac{d}{dt}\widehat{A}_t$ and understand this derivative in the sense of distributions: for any $t_1, t_2 \in [a,b]$, for any $\varphi \in C^m(M,E)$ and $q \in M$

$$\widehat{A}_{t_2}(\varphi)(q) - \widehat{A}_{t_1}(\varphi)(q) = \int\limits_{t_1}^{t_2} \frac{d}{dt}\widehat{A}_t\,(\varphi)(q)\, dt.$$

Remark 3.1. Let W be a C^m vector field and $\widehat{A}_t$ is absolutely continuous then $\widehat{A}_t \circ \widehat{W}$ is also absolutely continuous and $\dfrac{d}{dt}(\widehat{A}_t \circ W) = \dfrac{d}{dt}\widehat{A}_t \circ W$.

Here we discuss product rule for operator-valued functions $\widehat{P}_t$ and $\widehat{Q}_t$ in the sense of distributions for absolutely continuous operator-valued functions $\widehat{P}_t$ and $\widehat{Q}_t$ which are represented for any $t \in (a,b)$ as

$$\widehat{P}_t = \widehat{P}_{t_0} + \int\limits_{t_0}^{t} \frac{d}{d\tau}\widehat{P}_\tau \; d\tau, \quad \widehat{Q}_t = \widehat{Q}_{t_0} + \int\limits_{t_0}^{t} \frac{d}{d\tau}\widehat{Q}_\tau \; d\tau.$$

The following product rule for product $\widehat{P}_t \circ \widehat{Q}_t$ was proved in [10] under general assumptions which are satisfied if these operator-valued functions are solutions of the operator integral equations (3.6) with locally integrable vector fields.

Theorem 3.1. *Let absolutely continuous operator-valued function $\widehat{P}_t$ and $\widehat{Q}_t$ be solutions of operator integral equations. Then $\widehat{P}_t \circ \widehat{Q}_t$ is absolutely continuous and for any $t_1,\ t_2$ in (a,b)*

$$\int\limits_{t_1}^{t_2} \frac{d}{dt}\Big(\widehat{P}_t \circ \widehat{Q}_t\Big)\, dt = \int\limits_{t_1}^{t_2} \Big(\frac{d}{dt}\widehat{P}_t \circ \widehat{Q}_t + \widehat{P}_t \circ \frac{d}{dt}\widehat{Q}_t\Big)\, dt.$$

Let V be a vector field and $F: M \to M$ be a C^m diffeomorphism. Following [3] we define the operator $\mathrm{Ad}(\widehat{F}) : \widehat{V} \mapsto \widehat{F} \circ \widehat{V} \circ \widehat{F^{-1}}$.

Recall that the Lie bracket $[V, W]$ of vector fields V and W is the vector field whose operator representation has form $\widehat{[V,W]} = \widehat{V} \circ \widehat{W} - \widehat{W} \circ \widehat{V}$.

It makes sense (as in [3]) to define an operator $\mathrm{ad}(\widehat{V}_t)$ by $\mathrm{ad}(V_t) \circ \widehat{W} := \big[\widehat{V}_t, \widehat{W}\big]$.

We recall that in general case of measurable in t vector-field V_t we defined $\widehat{P}_{t_0,t}$ which satisfies the integral operator equation

$$\widehat{P}_{t_0,t} = \widehat{Id} + \int\limits_{t_0}^{t} \widehat{P}_{t_0,\tau} \circ \widehat{V}_\tau \, d\tau \tag{3.7}$$

and which is the unique absolutely continuous solution of this equation or the solution of the differential equation

$$\frac{d}{dt}\widehat{P}_{t_0,t} = \widehat{P}_{t_0,t} \circ \widehat{V}_t, \quad \widehat{P}_{t_0,t_0} = \widehat{Id}, \tag{3.8}$$

in sense of distributions. The justification of this fact is based on the relation of $\widehat{P}_{t_0,t}$ to the Carathéodory solutions of *ordinary* differential equation (3.4).

Now we consider the differential operator equation

$$\frac{d}{dt}\widehat{Q}_{t_0,t} = -\widehat{V}_t \circ \widehat{Q}_{t_0,t}, \quad \widehat{Q}_{t_0,t_0} = \widehat{Id}. \tag{3.9}$$

Note that this operator equation, even in the case $M = \mathbb{R}^n$, is related to some first-order linear *partial* differential equation.

The following result [10] states that for a locally integrable bounded C^m vector field V_t there exists a solution $\widehat{Q}_{t_0,t}$ of (3.9) in the sense of distributions. Moreover we have a representation of $\widehat{Q}_{t_0,t}$ in terms of a solution of the equation of the type (3.7). It can be easily proved by using the product rule from Theorem 3.1.

Proposition 3.1. *Let V_t be a locally integrable bounded C^m vector field. Then absolutely continuous operator-valued solutions $\widehat{P}_{t_0,t}$ and $\widehat{Q}_{t_0,t}$ of differential equations* (3.8) *and* (3.9) *exist, are unique and*

$$\widehat{Q}_{t_0,t} = (\widehat{P}_{t_0,t})^{-1}.$$

We also use the product rule from Theorem 3.1 and the previous Proposition 3.1 to prove the following important fact which is used in this paper.

Proposition 3.2. *Let V_t be locally integrable bounded C^m-smooth vector field and $\widehat{P}_{t_0,t}$ be an absolutely continuous solution of* (3.7). *Then for any C^m-smooth vector field W the operator-valued function $t \to \mathrm{Ad}(\widehat{P}_{t_0,t}) \circ \widehat{W}$ is absolutely continuous and satisfies the following equation in the sense of distributions:*

$$\frac{d}{dt}\mathrm{Ad}(\widehat{P}_{t_0,t}) \circ \widehat{W} = \mathrm{Ad}(\widehat{P}_{t_0,t}) \circ \mathrm{ad}(V_t) \circ \widehat{W}. \tag{3.10}$$

Method of variation of parameters can also be easy applied to the operator differential equation in the sense of distributions

$$\frac{d}{dt}\widehat{S}_{t_0,t} = \widehat{S}_{t_0,t} \circ (\widehat{V}_t + \widehat{W}_t), \quad \widehat{S}_{t_0,t_0} = \widehat{Id}. \tag{3.11}$$

Namely, we have the following (see [10]).

Proposition 3.3. *Let V_t, W_t be locally integrable bounded C^m-smooth vector fields. Then a solution of* (3.11) *can be represented in the form*

$$\widehat{S}_{t_0,t} = \widehat{C}_{t_0,t} \circ \widehat{P}_{t_0,t},$$

where $\widehat{P}_{t_0,t}$ *is the solution of the differential equation* (3.8) *and* $\widehat{C}_{t_0,t}$ *is a solution of the differential equation*

$$\frac{d}{dt}\widehat{C}_{t_0,t} = \widehat{C}_{t_0,t} \circ \operatorname{Ad}\widehat{P}_{t_0,t} \circ \widehat{W}_t, \quad \widehat{C}_{t_0,t_0} = \widehat{Id}. \tag{3.12}$$

From now on we use the following notation for brevity: $\widehat{P}_t := \widehat{P}_{0,t}$. We also use the following concept of the operator $\widehat{O}$.

Definition 3.4. Parametric family of linear operators $\widehat{O}(\alpha) : C^m(M;E) \to E$ is called *Big O of α at point* $q_0 \in M$ if for any $\psi \in C^m(M;E)$ there exists a neighbourhood $\mathcal{O}$ of q_0, a constant K such that for all small $\alpha > 0$

$$\|\widehat{O}(\alpha)\psi(q)\| < K\alpha \quad \forall\, q \in \mathcal{O}.$$

4. Dynamical Systems on Banach Manifolds and Wilcox Formula

Consider a parametric family of dynamical systems on Banach manifold M

$$q'(t) = V(t, q(t), \alpha), \tag{4.1}$$

where $q' := \dfrac{d}{dt}q$ and α is a scalar parameter.

As we know we can relate dynamical system (4.1) to the operator differential equation

$$\frac{d}{dt}\widehat{P}_t^\alpha = \widehat{P}_t^\alpha \circ \widehat{V}_t^\alpha, \quad \widehat{P}_0^\alpha = \widehat{Id}, \tag{4.2}$$

where $\widehat{P}_t^\alpha$ is a flow of diffeomorphisms corresponding the dynamical system (4.1) and the operator $\widehat{V}_t^\alpha$ corresponding to the vector field $V(t,q,\alpha)$.

Here we discuss assumptions on vector fields V_t^α which imply a generalization of Wilcox formula for the system (4.1).

Assumption 4.1. Let for given α and all small $\Delta\alpha$, all $q \in M$ and almost all t we have

$$V(t, q, \alpha + \Delta\alpha) = V(t, q, \alpha) + \Delta\alpha W(t, q) + U(t, q, \alpha, \Delta\alpha), \tag{4.3}$$

where for any t_1, t_2

$$\lim_{\Delta\alpha \to 0} \frac{1}{\Delta\alpha} \int_{t_1}^{t_2} \|\psi_*(q')U(s, q', \alpha, \Delta\alpha)\|\, ds = 0 \tag{4.4}$$

uniformly with respect to q' from some neighbourhood of q.

We use notation $\dfrac{\partial}{\partial\alpha}V(t,q,\alpha)$ for $W(t,q)$ in (4.3). Then the following generalized Wilcox formulas have a remarkable resemblance with the original Wilcox formula for operator exponentials (2.11).

Theorem 4.1. *Under Assumptions* 4.1 *operator-valued function* $\alpha \to \widehat{P}_t^\alpha$ *is differentiable and we have the following two representations for its derivative*:

$$\frac{\partial}{\partial\alpha}\widehat{P}_t^\alpha = \int_0^t \operatorname{Ad}(\widehat{P}_s^\alpha) \circ \frac{\partial}{\partial\alpha}\widehat{V}_s^\alpha\, ds \circ \widehat{P}_t^\alpha, \tag{4.5}$$

$$\frac{\partial}{\partial\alpha}\widehat{P}_t^\alpha = \widehat{P}_t^\alpha \circ \int_0^t \operatorname{Ad}(\widehat{P}_{t,s}^\alpha) \circ \frac{\partial}{\partial\alpha}\widehat{V}_s^\alpha\, ds. \tag{4.6}$$

Proof. We have from the Product Rule and (4.2) that

$$\widehat{P}_t^{\alpha+\Delta\alpha} \circ \left(\widehat{P}_t^{\alpha}\right)^{-1} - \mathrm{Id} = \int_0^t \frac{d}{ds} \widehat{P}_s^{\alpha+\Delta\alpha} \circ \left(\widehat{P}_s^{\alpha}\right)^{-1} \, ds$$

$$= \int_0^t \left(\widehat{P}_s^{\alpha+\Delta\alpha} \circ \widehat{V}_s^{\alpha+\Delta\alpha} \circ \left(\widehat{P}_s^{\alpha}\right)^{-1} - \widehat{P}_s^{\alpha+\Delta\alpha} \circ \widehat{V}_s^{\alpha} \left(\widehat{P}_s^{\alpha}\right)^{-1} \right) ds.$$

It follows from this relation and (4.3) that

$$\widehat{P}_t^{\alpha+\Delta\alpha} \circ \left(\widehat{P}_t^{\alpha}\right)^{-1} - \mathrm{Id} = \Delta\alpha \int_0^t \widehat{P}_s^{\alpha} \circ \widehat{W}_s \left(\widehat{P}_s\right)^{-1} \, ds \circ \widehat{P}_t^{\alpha} + \int_0^t \widehat{R}_s \, ds, \tag{4.7}$$

where $\widehat{R}_s = \widehat{U}_s + \Delta\alpha \left(\widehat{P}_s^{\alpha+\Delta\alpha} - \widehat{P}_s^{\alpha}\right) \circ \widehat{W}_s \circ \left(\widehat{P}_s^{\alpha}\right)^{-1}$. Then it follows from (4.4) and (4.7) that

$$\widehat{P}_t^{\alpha+\Delta\alpha} = \widehat{P}_t^{\alpha} + \Delta\alpha \int_0^t \mathrm{Ad}(\widehat{P}_s^{\alpha}) \circ \widehat{W}_s \, ds \circ \widehat{P}_s^{\alpha} + \hat{o}_t(\Delta\alpha).$$

This representation implies that $\alpha \to \widehat{P}_t^{\alpha}$ is differentiable and (4.5) holds.

To derive (4.6) from (4.5) we use semigroup properties of diffeomorphisms flows

$$\widehat{P}_s^{\alpha} = \widehat{P}_t^{\alpha} \circ \widehat{P}_{t,s}^{\alpha}, \quad \left(\widehat{P}_s^{\alpha}\right)^{-1} \circ \widehat{P}_t^{\alpha} = \left(\widehat{P}_{t,s}^{\alpha}\right)^{-1}.$$

5. Approximation of Dynamical Systems on Banach Manifolds

In this Section we use a generalized Wilcox formula (4.6) to obtain some useful approximations of solutions of the differential equation (1.1) by solutions of some more structured differential equation for small and large t.

It seems that such approximations can be applied to singular optimal control problems, to problems of controllability and stabilizability of nonlinear control systems etc.

Let α be a scalar parameter and we consider a parametric family of dynamical systems on the manifold

$$\frac{d}{dt} \, q = \alpha V_t(q), \tag{5.1}$$

where $V_t(q)$ is the vector field in (1.1). Thus, we have in (4.1) $V_t(q,\alpha) := \alpha V_t(q)$.

Then the diffeomorphisms flow $\widehat{P}_t^{\alpha}$ for dynamical system with the vector filed (5.1) is described by the operator differential equation

$$\frac{d}{dt} \widehat{P}_t^{\alpha} = \widehat{P}_t^{\alpha} \circ \alpha \widehat{V}_t. \tag{5.2}$$

It follows from Wilcox formula (4.6) that the diffeomorphisms flow $\widehat{P}_t^{\alpha}$ also satisfies the following equation:

$$\frac{d}{d\alpha} \widehat{P}_t^{\alpha} = \widehat{P}_t^{\alpha} \circ \int_0^t \mathrm{Ad}(\widehat{P}_{t,s}^{\alpha}) \circ \widehat{V}_s \, ds. \tag{5.3}$$

Note that this equation is some operator functional-differential equation but we show below that its solution can be approximated by a solution of some operator differential equation for small values of α.

Let us define the vector field operator

$$\widehat{F}_t^\alpha := \sum_{k=1}^{m-1} \alpha^{k-1} \widehat{W}_{k,t}, \tag{5.4}$$

where

$$\widehat{W}_{1,t} := \int_0^t \widehat{V}_{t_1}^\alpha \, dt_1,$$

$$\widehat{W}_{k,t} := \int_0^t dt_1 \int_t^{t_1} dt_2 \int_t^{t_2} dt_3 \ldots \int_t^{t_{k-1}} dt_k \big[\widehat{V}_{t_k}, [\widehat{V}_{t_{k-1}}, \ldots, [\widehat{V}_{t_2}, \widehat{V}_{t_1}], \ldots,]\big], \quad k = 2, \ldots, m-1\,.$$

In the next result we demonstrate that the solution $\widehat{S}_t^\alpha$ of the following operator differential equation:

$$\frac{d}{d\alpha} \widehat{Z}_t^\alpha = \widehat{Z}_t^\alpha \circ \widehat{F}_t^\alpha, \quad \widehat{Z}_t^\alpha \Big|_{\alpha=0} = \widehat{Id}, \tag{5.5}$$

approximates the solution $\widehat{P}_t^\alpha$ of (5.3) (and of (5.2)) for α small enough, namely,

$$\widehat{P}_t^\alpha = \widehat{Z}_t^\alpha + \widehat{O}(\alpha^m). \tag{5.6}$$

Before its precise statement we write differential equation on the manifold corresponding to the operator differential equation (5.5)

$$\frac{d}{d\alpha} \, z = F_t^\alpha(z), \tag{5.7}$$

where

$$F_t^\alpha(z) := \sum_{k=1}^{m-1} \alpha^{k-1} W_{k,t}(z), \tag{5.8}$$

$$W_{1,t}(z) := \int_0^t V_{t_1}(z) \, dt_1,$$

$$W_{k,t}(z) := \int_0^t dt_1 \int_t^{t_1} dt_2 \int_t^{t_2} dt_3 \ldots \int_t^{t_{k-1}} dt_k \big[V_{t_k}, [\widehat{V}_{t_{k-1}}, \ldots, [V_{t_2}, V_{t_1}], \ldots,]\big](z), \quad k = 2, \ldots, m-1\,.$$

Assumption 5.1. For any $q_0 \in M$ there exists $\alpha > 0$, θ_0 and neighbourhood $\mathcal{O}$ of q_0 such that for any α solution $q(t)$ of (5.1) with imitial condition $q(0) \in \mathcal{O}$ exists on $[0, \theta_0)$.

The following Proposition's statement is the reformulation of the relation (5.5).

Proposition 5.1. *Under Assumptions* 4.1, 5.1 *for any* $q_0 \in M$*, function* $\psi \in C^m(M, E)$ *there exists a constant* K *such that for all sufficiently small* α *and* $t \in [0, \theta_0)$ *and any solutions* $q\alpha(t)$ *of* (5.1) *and* $z^t(\alpha)$ *of* (5.7) *with initial conditions* $z(0) = q(0) \in \mathcal{O}$

$$\|\psi(q^\alpha(t)) - \psi(z^t(\alpha))\| < K\alpha^m. \tag{5.9}$$

Proof. We use extended Chronological Calculus to prove (5.5) (or equivalent (5.9)). Note that the vector field operator in (5.3) is written as

$$\int_0^t \mathrm{Ad}(\widehat{P}_{t,t_1}^\alpha) \circ \widehat{V}_{t_1} \, dt_1. \tag{5.10}$$

Then it follows from (3.10) that

$$\mathrm{Ad}(\widehat{P}^{\alpha}_{t,t_1}) \circ \widehat{W} = \widehat{W} + \int\limits_{t}^{t_1} dt_2 \mathrm{Ad}(\widehat{P}^{\alpha}_{t,t_2}) \circ \widehat{W}.$$

By using recursively this relation for (5.10) we obtain that $\int_0^t \mathrm{Ad}(\widehat{P}^{\alpha}_{t,t_1}) \circ \widehat{V}_{t_1}\ dt_1 = \widehat{F}^{\alpha}_t + \alpha^{m-1}\widehat{R}^{\alpha}_t$, where $\widehat{F}^{\alpha}_t$ is defined in (5.4) and

$$\widehat{R}^{\alpha}_t := \int\limits_0^t dt_1 \int\limits_t^{t_1} dt_2 \int\limits_t^{t_2} dt_3 \dots \int\limits_t^{t_{m-1}} dt_m \mathrm{Ad}(\widehat{P}^{\alpha}_{t,t_m}) \circ \big[\widehat{V}_{t_m}, [\widehat{V}_{t_{k-1}}, \dots, [\widehat{V}_{t_2}, \widehat{V}_{t_1}], \dots,]\big].$$

Thus, it follows from (5.5) that $\widehat{P}^{\alpha}_t$ satisfies the equation $\dfrac{d}{d\alpha}\widehat{P}^{\alpha}_t = \widehat{P}^{\alpha}_t \circ (\widehat{F}^{\alpha}_t + \alpha^{m-1}\widehat{R}^{\alpha}_t)$. We use the method of variation of parameters to find its solution in the form $\widehat{P}^{\alpha}_t = \widehat{C}_t \circ \widehat{Z}^{\alpha}_t$ to obtain a formula similar to (3.12) that

$$\widehat{C}^{\alpha}_t = \widehat{Id} + \int\limits_0^{\alpha} \widehat{C}^{\sigma}_t \circ \mathrm{Ad}(\widehat{Z}^{\sigma}_t) \circ \sigma^{m-1}\widehat{R}^{\sigma}_t\ d\sigma.$$

From this representation we obtain (5.6) which implies the assertion of the Proposition 5.1. □

Approximation of solutions of dynamical systems for small time. Consider the solution $q(t)$ of the initial-value problem

$$\frac{d}{dt}\ q(t) = V_t(q(t)), \quad q(0) = q_0. \tag{5.11}$$

Here we use Proposition 5.1 to show that for any small T the value $q(T)$ can be approximated by the value $z(T)$ of the solution $z(\alpha)$ of the following structured differential equation:

$$\frac{d}{d\alpha} z(\alpha) = G^{\alpha}_T(z(\alpha)), \quad z(0) = q_0, \tag{5.12}$$

where

$$G^{\alpha}_T(z) := \sum_{k=1}^{m-1} \alpha^{k-1} \int\limits_0^1 d\tau_1 \int\limits_1^{\tau_1} d\tau_2 \dots \int\limits_1^{\tau_{k-1}} d\tau_k \big[V_{T\tau_k}, [V_{T\tau_{k-1}}, \dots, [V_{T\tau_2}, V_{T\tau_1}], \dots,]\big](z).$$

Theorem 5.1. *Let V_t in (5.11) be locally bounded. Then for any $q_0 \in M$, function $\psi \in C^m(M; E)$ there exists a constant K such that for any $T > 0$ small enough*

$$\|\psi(q(T)) - \psi(z(T))\| < KT^m. \tag{5.13}$$

Proof. We fix small $T > 0$, define $\alpha_* := T$ and make substitution $t = T\tau$, $\tilde{q}(\tau) := q(T\tau)$. Then

$$\frac{d}{d\tau}\tilde{q}(\tau) = \alpha_* V_{T\tau}(\tilde{q}(\tau)), \quad \tilde{q}(0) = q_0.$$

Note that this is a particular case of the differential equation (5.1) and $\tilde{q}(1) = q(T)$.

Then it follows from the Proposition 5.1 that for any function $\psi \in C^m(M; E)$ there exists a constant K such that for all small $T > 0$ $\|\psi(\tilde{q}(1)) - \psi(z(\alpha_*))\| < K\alpha_*^m$, where $z(\alpha)$ is a solution of the differential equation (5.12).

But this inequality is the relation (5.13). □

Approximation of solutions of dynamical systems for large time. Consider the solution $q(t)$ of the initial-value problem (5.11).

Here we use Proposition 5.1 to show that for any large $T > 0$ the value $q(T)$ can be approximated by the value $z(T^2)$ of the solution $z(\alpha)$ of the structured differential equation (5.12), where $G_T^\alpha(z)$ is given by the expression

$$\sum_{k=1}^{m-1} \alpha^{k-1} \int\limits_0^{T^2} d\tau_1 \int\limits_{T^2}^{\tau_1} d\tau_2 \ldots \int\limits_{T^2}^{\tau_{k-1}} d\tau_k \big[V_{\tau_k/T}, [V_{\tau_{k-1}/T}, \ldots, [V_{\tau_2/T}, V_{\tau_1/T}], \ldots,]\big](z). \tag{5.14}$$

Theorem 5.2. *Let for any $q_0 \in M$ solution $q(t)$ of* (5.11) *exist on* $(0, +\infty)$. *Then for any function $\psi \in C^m(M; E)$ there exists a constant K such that for any $T > 0$ large enough we have for solution $z(\alpha)$ of* (5.12), (5.14)

$$\|\psi(q(T)) - \psi(z(T^2))\| < \frac{K}{T^m}. \tag{5.15}$$

Proof. We fix large $T > 0$, define $\alpha_* := 1/T$ and make substitution $t = \tau/T$, $\tilde{q}(\tau) := q(\tau/T)$. Then

$$\frac{d}{d\tau}\tilde{q}(\tau) = \alpha_* V_{\tau/T}(\tilde{q}(\tau)), \quad \tilde{q}(0) = q_0.$$

Note that this is a particular case of the differential equation (5.1) and $\tilde{q}(T^2) = q(T)$.

Then it follows from the Proposition 5.1 that for any function $\psi \in C^m(M; E)$ there exists a constant K such that for all small $T > 0$ $\|\psi(\tilde{q}(T^2)) - \psi(z(\alpha_*))\| < K\alpha_*^m$, where $z(\alpha)$ is a solution of the differential equation (5.12)–(5.14).

But this inequality is the relation (5.15). □

Acknowledgement

The author is grateful to Robert Kipka of Lake Superior State University and Nina N. Subbotina of Krasovskii Institute of Mathematics and Mechanics for stimulating discussions.

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Received July 20, 2021
Revised August 11, 2021
Accepted August 23, 2021

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Cite this article as: Yuri S. Ledyaev. Wilcox Formula for Vector Fields on Banach Manifolds, *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2021, vol. 27, no. 3, pp. 271–285 .