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ON THE ROBUSTNESS PROPERTY OF A CONTROL SYSTEM DESCRIBED BY AN URYSOHN TYPE INTEGRAL EQUATION

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In this paper a control system described by an Urysohn type integral equation with an integral constraint on the control functions is studied. It is assumed that the system is nonlinear with respect to the state vector and is affine with respect to the control vector. The control functions are chosen from a closed ball of the space L_p ($p > 1$) with radius r. It is proved that the set of trajectories of the control system generated by all admissible control functions is Lipschitz continuous with respect to r and is continuous with respect to p as a set valued map. It is shown that the system's trajectory is robust with respect to the full consumption of the remaining control resource and every trajectory can be approximated by a trajectory generated by the control function with full control resource consumption.

Keywords: integral equation, control system, integral constraint, set of trajectories, robustness.

Н. Гусейин, А. Гусейин, Х. Г. Гусейнов. О свойстве робастности управляемой системы, описываемой интегральным уравнением типа Урысона.

В данной работе исследуется управляемая система, описываемая интегральным уравнением типа Урысона с интегральным ограничением на управляющие функции. Предполагается, что система нелинейна по фазовому вектору и аффинна по управляющему вектору. Управления выбираются из замкнутого шара пространства L_p ($p > 1$) радиуса r с центром в начале координат. Доказано, что множество траекторий управляемой системы, отвечающих всем допустимым управлениям, липшицево по r и непрерывно по p как многозначное отображение. Показано, что траектория системы робастна относительно полного использования оставшегося ресурса управления и любую траекторию системы можно аппроксимировать траекторией, соответствующей управлению с полным использованием ресурса.

Ключевые слова: интегральное уравнение, управляемая система, интегральное ограничение, множество траекторий, робастность.

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Introduction

Integral equations are an adequate tool to describe different processes arising in physics, economy, biology, medicine, etc. (see, e.g., [4; 5; 12] and references therein). It is known that some of solution concepts for initial and boundary value problems for differential equations can be reduced to solution notions of appropriate integral equations (see, e.g., [5; 13]). It occurs that some processes described via integral equations have external influences called control efforts. Integral constraints on the control functions are inevitable if the control resource, such as energy, fuel, finance, etc., is exhausted by consumption (see, e.g., [3; 7; 14–16]). The various topological properties and approximate construction methods of the set of trajectories of control systems described by integral equations with an integral constraint on the control functions are investigated in [2; 8–11]. In this paper, the set of trajectories of a control system described by an Urysohn type integral equation is considered. Admissible control functions are chosen from a closed ball of the space L_p, $p > 1$, centered at the origin with radius r. The dependence of the set of trajectories on the parameters r and p is studied and the robustness of a trajectory with respect to a fast consumption of the remaining control resource is discussed.

The paper is organized as follows. In Section 2 the basic conditions and auxiliary propositions which are used in the following arguments are given. In Section 3 it is proved that the set of trajectories is a Lipschitz continuous set valued map with respect to the parameter r, which characterizes a bound on the control resource (Theorem 1). In Section 4 it is shown that the set of trajectories depends continuously on p (Theorem 2). In Section 5 it is proved that the consumption of the remaining control resource on a domain with sufficiently small measure does not cause an essential change in the trajectory of the system (Theorem 3). It is shown that the set of trajectories generated by a full consumption of the control resource is dense in the set of trajectories generated by all admissible control functions (Theorem 4).

1. Description of the System

Consider a control system described by the Urysohn type integral equation

$$x(t) = f(t, x(t)) + \lambda \int_a^b \left[K_1(t, s, x(s)) + K_2(t, s, x(s))\, u(s)\right] ds, \tag{1.1}$$

where $x(s) \in \mathbb{R}^n$ is the state vector, $u(s) \in \mathbb{R}^m$ is the control vector, $t \in [a, b]$, and $\lambda > 0$ is a given number.

For $p > 1$ and $r \geq 0$, define $V_{p,r} = \{u(\cdot) \in L_p([a, b], \mathbb{R}^m) \colon \|u(\cdot)\|_p \leq r\}$, where $L_p([a, b], \mathbb{R}^m)$ is the space of Lebesgue measurable functions $u(\cdot) \colon [a, b] \to \mathbb{R}^m$ such that $\|u(\cdot)\|_p < +\infty$, $\|u(\cdot)\|_p = \left(\int_a^b \|u(t)\|^p \, dt\right)^{1/p}$, and $\|\cdot\|$ denotes the Euclidean norm. It is assumed that the functions and the number λ given in equation (1.1) satisfy the following conditions.

2.A. The vector functions $f(\cdot, \cdot) \colon [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $K_1(\cdot, \cdot, \cdot) \colon [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ and the matrix function $K_2(\cdot, \cdot, \cdot) \colon [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are continuous.

2.B. There exist $\gamma_0 \in [0, 1)$, $\gamma_1 \geq 0$, and $\gamma_2 \geq 0$ such that

$$\|f(t, x_1) - f(t, x_2)\| \leq \gamma_0 \|x_1 - x_2\|$$

for every $(t, x_1) \in [a, b] \times \mathbb{R}^n$, $(t, x_2) \in [a, b] \times \mathbb{R}^n$ and

$$\|K_1(t, s, x_1) - K_1(t, s, x_2)\| \leq \gamma_1 \|x_1 - x_2\|,$$

$$\|K_2(t, s, x_1) - K_2(t, s, x_2)\| \leq \gamma_2 \|x_1 - x_2\|$$

for every $(t, s, x_1) \in [a, b] \times [a, b] \times \mathbb{R}^n$, $(t, s, x_2) \in [a, b] \times [a, b] \times \mathbb{R}^n$.

2.C. There exist $p_* > 1$ and $r_* > 0$ such that $0 \leq \lambda\left(\gamma_1 (b - a) + \gamma_2 r_* (b - a)^{(p_* - 1)/p_*}\right) < 1 - \gamma_0$.

Define

$$L(\lambda; p, r) = \gamma_0 + \lambda\left[\gamma_1 (b - a) + \gamma_2 r\, (b - a)^{(p-1)/p}\right]. \tag{1.2}$$

Condition **2.C** implies that $L(\lambda; p_*, r_*) < 1$. From (1.2) it follows that there exist $\tau_1 > 0$ and $\tau_2 > 0$ such that $p_* - \tau_1 > 1$ and $L(\lambda; p, r) < 1$ for every $p \in [p_* - \tau_1, p_* + \tau_1]$ and $r \in [0, r_* + \tau_2]$. From now on it will be assumed that $p \in [p_* - \tau_1, p_* + \tau_1]$ and $r \in [0, r_* + \tau_2]$. Define

$$h_* = \max\left\{(b - a)^{(p-1)/p} \colon p \in [p_* - \tau_1, p_* + \tau_1]\right\}, \tag{1.3}$$

$$L_*(\lambda) = \gamma_0 + \lambda\left[\gamma_1 (b - a) + \gamma_2 (r_* + \tau_2) h_*\right]. \tag{1.4}$$

It is obvious that $0 < L_*(\lambda) < 1$.

Now, let us define a trajectory of system (1.1) generated by an admissible control function $u(\cdot) \in V_{p,r}$. A continuous function $x(\cdot) \colon [a, b] \to \mathbb{R}^n$ satisfying the integral equation (1.1) for every

$t \in [a,b]$ is said to be a trajectory of system (1.1) generated by the admissible control function $u(\cdot) \in V_{p,r}$. By virtue of [8], every admissible control function $u(\cdot)$ generates a unique trajectory of system (1.1). The set of trajectories of system (1.1) generated by all admissible control functions $u(\cdot) \in V_{p,r}$ is denoted by $\mathbf{Z}_{p,r}$. The set $\mathbf{Z}_{p,r}$ is called the set of trajectories of system (1.1). It is obvious that $\mathbf{Z}_{p,r} \subset C([a,b];\mathbb{R}^n)$, where $C([a,b];\mathbb{R}^n)$ is the space of continuous functions $x(\cdot)\colon [a,b] \to \mathbb{R}^n$ with the norm $\|x(\cdot)\|_C = \max\{\|x(t)\| : t \in [a,b]\}$. For $t \in [a,b]$ we set

$$\mathbf{Z}_{p,r}(t) = \{x(t) \in \mathbb{R}^n : x(\cdot) \in \mathbf{Z}_{p,r}\}. \tag{1.5}$$

According to [8], $\mathbf{Z}_{p,r}$ and $\mathbf{Z}_{p,r}(t)$, $t \in [a,b]$, are nonempty compact sets and the set valued map $t \to \mathbf{Z}_{p,r}(t)$, $t \in [a,b]$, is continuous in the Hausdorff metric. Moreover, it is possible to show that there exists $c_* > 0$ such that

$$\|x(\cdot)\|_C \le c_* \tag{1.6}$$

for every $x(\cdot) \in \mathbf{Z}_{p,r}$, $p \in [p_* - \tau_1, p_* + \tau_1]$, and $r \in [0, r_* + \tau_2]$. Define

$$M_2 = \max\{\|K_2(t,s,x)\| : t \in [a,b],\ s \in [a,b],\ \|x\| \le c_*\}, \tag{1.7}$$

where c_* is given by (1.6).

2. Continuity with Respect to r

In this section the Lipschitz continuity of the set valued map $r \to \mathbf{Z}_{p,r}$, $r \in [0, r_* + \tau_2]$, is proved. The Hausdorff distance between sets $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^n$ is denoted by $H_n(P,Q)$, and the Hausdorff distance between sets $Y \subset C([a,b];\mathbb{R}^n)$ and $E \subset C([a,b];\mathbb{R}^n)$ is denoted by $H_C(Y,E)$ (see, e.g., [1]). We set

$$B_C(1) = \{x(\cdot) \in C([a,b];\mathbb{R}^n) :\ \|x(\cdot)\|_C \le 1\}, \tag{2.1}$$

$$R_* = \frac{\lambda M_2 h_*}{1 - L_*(\lambda)}, \tag{2.2}$$

where h_*, $L_*(\lambda)$, and M_2 are defined by (1.3), (1.4), and (1.7) respectively.

Theorem 1. *For each fixed $p \in [p_* - \tau_1, p_* + \tau_1]$ the inequality*

$$H_C(\mathbf{Z}_{p,r_2}, \mathbf{Z}_{p,r_1}) \le R_* |r_2 - r_1| \tag{2.3}$$

is satisfied for every $r_2 \in [0, r_* + \tau_2]$ and $r_1 \in [0, r_* + \tau_2]$.

Proof. Let $r_2 \ne 0$. Choose an arbitrary $x_*(\cdot) \in \mathbf{Z}_{p,r_2}$ generated by an admissible control function $u_*(\cdot) \in V_{p,r_2}$. Define

$$v_*(t) = \frac{r_1}{r_2} u_*(t), \quad t \in [a,b]. \tag{2.4}$$

If $r_2 = 0$, then V_{p,r_2} includes a unique control function $u(t) = 0$ for almost all $t \in [a,b]$. In this case $v_*(\cdot)$ is chosen as an arbitrarily function from V_{p,r_1}. It is obvious that $v_*(\cdot) \in V_{p,r_1}$. From (1.3), (2.4), the inclusion $u_*(\cdot) \in V_{p,r_2}$, and Hölder's inequality, we get

$$\int_a^b \|u_*(s) - v_*(s)\|\, ds \le \frac{|r_1 - r_2|}{r_2} (b-a)^{(p-1)/p} \|u_*(\cdot)\|_p \le h_* |r_1 - r_2|. \tag{2.5}$$

Let $y_*(\cdot)\colon [a,b] \to \mathbb{R}^n$ be the trajectory of system (1.1) generated by the admissible control function $v_*(\cdot)$. Then $y_*(\cdot) \in \mathbf{Z}_{p,r_1}$, and from (1.1), (1.4), (1.6), (1.7), (2.5), and Condition **2.B** we get

$$\begin{aligned}\|x_*(t) - y_*(t)\| &\leq \gamma_0 \|x_*(t) - y_*(t)\| + \lambda \int\limits_a^b [\gamma_1 + \gamma_2 \|u_*(s)\|] \|x_*(s) - y_*(s)\| \, ds \\ &+ \lambda M_2 h_* |r_2 - r_1| \\ &\leq \left[\gamma_0 + \lambda\big(\gamma_1(b-a) + \gamma_2(b-a)^{(p-1)/p} r_2\big)\right] \|x_*(\cdot) - y_*(\cdot)\|_C + \lambda M_2 h_* |r_2 - r_1| \\ &\leq L_*(\lambda) \|x_*(\cdot) - y_*(\cdot)\|_C + \lambda M_2 h_* |r_2 - r_1|\end{aligned}$$

for every $t \in [a,b]$. From this inequality and (2.2), we obtain

$$\|x_*(\cdot) - y_*(\cdot)\|_C \leq R_* |r_2 - r_1| . \tag{2.6}$$

Thus, we have proved that for arbitrarily chosen $x_*(\cdot) \in \mathbf{Z}_{p,r_2}$ there exists $y_*(\cdot) \in \mathbf{Z}_{p,r_1}$ such that inequality (2.6) is satisfied. This means that

$$\mathbf{Z}_{p,r_2} \subset \mathbf{Z}_{p,r_1} + R_* |r_2 - r_1| B_C(1), \tag{2.7}$$

where $B_C(1)$ is defined by (2.1). Similarly, it is possible to show that

$$\mathbf{Z}_{p,r_1} \subset \mathbf{Z}_{p,r_2} + R_* |r_2 - r_1| B_C(1). \tag{2.8}$$

Inclusions (2.7) and (2.8) imply the validity of inequality (2.3). □

From Theorem 1 we conclude that for each fixed $p \in [p_* - \tau_1, p_* + \tau_1]$ the set valued map $r \to \mathbf{Z}_{p,r}$, $r \in [0, r_* + \tau_2]$, is Lipschitz continuous in the Hausdorff metric, which implies the validity of the following corollary.

Corollary 1. *For each fixed $p \in [p_* - \tau_1, p_* + \tau_1]$ the inequality*

$$H_n\left(\mathbf{Z}_{p,r_2}(t), \mathbf{Z}_{p,r_1}(t)\right) \leq R_* |r_2 - r_1|$$

is satisfied for every $r_2 \in [0, r_* + \tau_2]$, $r_1 \in [0, r_* + \tau_2]$, and $t \in [a,b]$, where $\mathbf{Z}_{p,r}(t)$ is defined by (1.5).

3. Continuity with Respect to p

In this section we establish the continuity of the set valued map $p \to \mathbf{Z}_{p,r}$, $p \in (p_* - \tau_1, p_* + \tau_1)$.

The Hausdorff distance between the sets $U \subset L_{p_1}([a,b], \mathbb{R}^m)$ and $W \subset L_{p_2}([a,b], \mathbb{R}^m)$ is denoted by $\mathcal{H}_1(U,W)$ and is defined as

$$\mathcal{H}_1(U,W) = \max\left\{ \sup_{x(\cdot)\in W} d_1(x(\cdot), U),\ \sup_{y(\cdot)\in U} d_1(y(\cdot), W) \right\},$$

where $d_1(x(\cdot), U) = \inf\left\{ \int_a^b \|x(s) - y(s)\| \, ds\colon y(\cdot) \in U \right\}$, $p_1 \in [1,\infty)$, $p_2 \in [1,\infty)$.

Theorem 2. *For each fixed $r \in [0, r_* + \tau_2]$ the set valued map $p \to \mathbf{Z}_{p,r}$, $p \in (p_* - \tau_1, p_* + \tau_1)$, is continuous in the Hausdorff metric.*

Proof. Let us choose an arbitrary $p_0 \in (p_* - \tau_1, p_* + \tau_1)$ and let $\varepsilon > 0$ be a given number. According to Theorem 3.6 from [6], for given $\varepsilon > 0$ there exists $\delta_1(\varepsilon, p_0) \in (0, \delta_*)$ such that the inequality

$$\mathcal{H}_1(V_{p,r}, V_{p_0,r}) < \varepsilon \tag{3.1}$$

holds for all $p \in (p_0 - \delta_1(\varepsilon, p_0), p_0 + \delta_1(\varepsilon, p_0))$, where $\delta_* = \min\{p_* + \tau_1 - p_0,\ p_0 + \tau_1 - p_*\}$. We will show that for every $p \in (p_0 - \delta_1(\varepsilon, p_0), p_0 + \delta_1(\varepsilon, p_0))$ the inequality

$$H_C\left(\mathbf{Z}_{p,r}, \mathbf{Z}_{p_0,r}\right) \leq \frac{\lambda M_2}{1 - L_*(\lambda)}\,\varepsilon \tag{3.2}$$

is verified, where $L_*(\lambda)$ and M_2 are defined by (1.4) and (1.7), respectively.

Choose arbitrary $p \in (p_0 - \delta_1(\varepsilon, p_0), p_0 + \delta_1(\varepsilon, p_0))$. Now let us choose arbitrary $y(\cdot) \in \mathbf{Z}_{p,r}$ generated by the admissible control function $v(\cdot) \in V_{p,r}$. According to (3.1), for $v(\cdot) \in V_{p,r}$ there exists $w(\cdot) \in V_{p_0,r}$ such that

$$\int_a^b \|v(s) - w(s)\|\, ds < \varepsilon. \tag{3.3}$$

Let $z(\cdot)$ be the trajectory generated by the admissible control function $w(\cdot) \in V_{p_0,r}$. Then $z(\cdot) \in \mathbf{Z}_{p_0,r}$ and from (1.1), (3.3), and Conditions **2.B** and **2.C** it follows that

$$\begin{aligned}
\|y(t) - z(t)\| &\leq \frac{\lambda}{1-\gamma_0}\int_a^b \left[\gamma_1 + \gamma_2\|v(s)\|\right]\|y(s) - z(s)\|\, ds + \frac{\lambda M_2}{1-\gamma_0}\int_a^b \|v(s) - w(s)\|\, ds \\
&\leq \frac{\lambda}{1-\gamma_0}\int_a^b \left[\gamma_1 + \gamma_2\|v(s)\|\right]\|y(s) - z(s)\|\, ds + \frac{\lambda M_2}{1-\gamma_0}\varepsilon \\
&\leq \frac{\lambda}{1-\gamma_0}\left(\gamma_1(b-a) + \gamma_2 r(b-a)^{(p-1)/p}\right)\|y(\cdot) - z(\cdot)\|_C + \frac{\lambda M_2}{1-\gamma_0}\varepsilon \\
&\leq \frac{L_*(\lambda) - \gamma_0}{1-\gamma_0}\|y(\cdot) - z(\cdot)\|_C + \frac{\lambda M_2}{1-\gamma_0}\varepsilon
\end{aligned}$$

for every $t \in [a, b]$; hence,

$$\|y(\cdot) - z(\cdot)\|_C \leq \frac{\lambda M_2}{1 - L_*(\lambda)}\varepsilon. \tag{3.4}$$

Thus we conclude that for an arbitrarily chosen $y(\cdot) \in \mathbf{Z}_{p,r}$ there exists $z(\cdot) \in \mathbf{Z}_{p_0,r}$ such that inequality (3.4) is satisfied. This means that

$$\mathbf{Z}_{p,r} \subset \mathbf{Z}_{p_0,r} + \frac{\lambda M_2}{1 - L_*(\lambda)}\varepsilon B_C(1), \tag{3.5}$$

where $B_C(1)$ is defined by (2.1).

Analogously, it is possible to show that

$$\mathbf{Z}_{p_0,r} \subset \mathbf{Z}_{p,r} + \frac{\lambda M_2}{1 - L_*(\lambda)}\varepsilon B_C(1). \tag{3.6}$$

From (3.5) and (3.6) we obtain the validity of (3.2), which proves the theorem. □

Corollary 2. *For each fixed* $r \in [0, r_* + \tau_2]$, *the set valued map* $p \to \mathbf{Z}_{p,r}(t)$, $p \in (p_* - \tau_1, p_* + \tau_1)$, *is continuous in the Hausdorff metric uniformly in* $t \in [a, b]$. *Here the set* $\mathbf{Z}_{p,r}(t)$ *is defined by* (1.5).

4. Robustness with Respect to Resource Consumption

In this section we will establish that every trajectory of the system is robust with respect to the fast consumption of the remaining control resource; i.e. the consumption of the remaining control resource on a domain with sufficiently small measure causes a small change of the system's trajectory.

Theorem 3. *Suppose that $\varepsilon > 0$ is a given number, $p \in (p_* - \tau_1, p_* + \tau_1)$, $x(\cdot) \in \mathbf{Z}_{p,r}$ is a trajectory of system (1.1) generated by an admissible control function $u(\cdot) \in V_{p,r}$, $\|u(\cdot)\|_p = r_* < r$, $E_* \subset [a,b]$ is a Lebesgue measurable set, a control function*

$$w(t) = \begin{cases} u(t) & \text{if } t \in [a,b] \setminus E_*, \\ u_*(t) & \text{if } t \in E_* \end{cases} \tag{4.1}$$

is such that $\|w(\cdot)\|_p = r$, and $z(\cdot) \in \mathbf{Z}_{p,r}$ is the trajectory of the system (1.1) generated by the admissible control function $w(\cdot) \in V_{p,r}$. If

$$\mu(E_*) \leq \Big[\frac{1 - L_*(\lambda)}{2\lambda r M_2}\, \varepsilon\Big]^{p/(p-1)}, \tag{4.2}$$

then

$$\|x(\cdot) - z(\cdot)\|_C \leq \varepsilon, \tag{4.3}$$

where $\mu(E_*)$ denotes the Lebesgue measure of the set E_* and $L_*(\lambda)$ and M_2 are defined by formulas (1.4) and (1.7), respectively.

Proof. According to Conditions **2.B** and **2.C**, inclusions $u(\cdot) \in V_{p,r}$ and $w(\cdot) \in V_{p,r}$, Hölder's inequality, and (4.1), we have

$$\begin{aligned} \|x(t) - z(t)\| &\leq \frac{\lambda}{1-\gamma_0} \int_a^b [\gamma_1 + \gamma_2 \|u(s)\|]\, \|x(s) - z(s)\|\, ds + \frac{\lambda M_2}{1-\gamma_0} \int_{E_*} \|u(s) - w(s)\|\, ds \\ &\leq \frac{\lambda}{1-\gamma_0} \int_a^b [\gamma_1 + \gamma_2 \|u(s)\|]\, ds \cdot \|x(\cdot) - z(\cdot)\|_C + \frac{2\lambda r M_2}{1-\gamma_0} [\mu(E_*)]^{(p-1)/p} \\ &\leq \frac{L_*(\lambda) - \gamma_0}{1-\gamma_0} \|x(\cdot) - z(\cdot)\|_C + \frac{2\lambda r M_2}{1-\gamma_0} [\mu(E_*)]^{(p-1)/p} \end{aligned}$$

for every $t \in [a,b]$. Therefore, it follows from (4.2) that

$$\|x(\cdot) - z(\cdot)\|_C \leq \frac{2\lambda r M_2}{1 - L_*(\lambda)} [\mu(E_*)]^{(p-1)/p} \leq \varepsilon.$$

The validity of inequality (4.3) is proved. □

As follows from Theorem 3, the consumption of the control resource in big quants on domains with sufficiently small measures is not an effective way to change the system's trajectory. We also find from Theorem 3 that if we have an excess of the control resource and we want to get rid of it, then, by consuming all the remaining control resource on a domain with sufficiently small measure, we can achieve a minor variation of the system's trajectory. Define

$$V_{p,r}^* = \big\{u(\cdot) \in L_p\big([a,b], \mathbb{R}^m\big) : \ \|u(\cdot)\|_p = r\big\},$$

and let $\mathbf{Z}_{p,r}^*$ be the set of trajectories of system (1.1) generated by all control functions $u(\cdot) \in V_{p,r}^*$.

Theorem 4. *The equality*

$$\mathrm{cl}\,\left(\mathbf{Z}^*_{p,r}\right) = \mathbf{Z}_{p,r}$$

is satisfied, where cl denotes the closure of a set.

Proof. Since $\mathbf{Z}_{p,r} \subset C\left([a,b]\,;\mathbb{R}^n\right)$ is a compact set (see [8]) and $\mathbf{Z}^*_{p,r} \subset \mathbf{Z}_{p,r}$, we have

$$\mathrm{cl}\,\left(\mathbf{Z}^*_{p,r}\right) \subset \mathbf{Z}_{p,r}. \tag{4.4}$$

Let us choose an arbitrarily number $\eta > 0$, and let $z(\cdot) \in \mathbf{Z}_{p,r}$ be a trajectory of system (1.1) generated by a control function $u(\cdot) \in V_{p,r}$ such that $\|u(\cdot)\|_p = \tilde{r} < r$. Choose an arbitrary Lebesgue measurable set $\Omega_* \subset [a,b]$ such that

$$\mu(\Omega_*) \le \left[\frac{1 - L_*(\lambda)}{2\lambda r M_2}\,\eta\right]^{p/(p-1)}, \tag{4.5}$$

where $L_*(\lambda)$ and M_2 are defined by (1.4) and (1.7), respectively.

Define a new control function, setting

$$v(t) = \begin{cases} u(t) & \text{if} \quad t \in [a,b] \setminus \Omega_*, \\ \left[\dfrac{r^p - \sigma^p}{\mu(\Omega_*)}\right]^{1/p} \xi & \text{if} \quad t \in \Omega_*, \end{cases} \tag{4.6}$$

where $\sigma^p = \displaystyle\int_{[a,b]\setminus\Omega_*} \|u(s)\|^p\,ds$, $\xi \in \mathbb{R}^m$, and $\|\xi\| = 1$. Using (4.6), it is not difficult to verify that $\|v(\cdot)\|_p = r$, and hence $v(\cdot) \in V^*_{p,r}$. Let $y(\cdot)$ be a trajectory of system (1.1) generated by the control function $v(\cdot) \in V^*_{p,r}$. Then $y(\cdot) \in \mathbf{Z}^*_{p,r}$, and, by inequality (4.5) and Theorem 3, we have $\|z(\cdot) - y(\cdot)\|_C \le \eta$. Since $z(\cdot) \in \mathbf{Z}_{p,r}$ is chosen arbitrarily, we obtain $\mathbf{Z}_{p,r} \subset \mathbf{Z}^*_{p,r} + \eta B_C(1)$.

Since $\eta > 0$ is chosen arbitrarily, the latter inclusion yields

$$\mathbf{Z}_{p,r} \subset \mathrm{cl}\,\left(\mathbf{Z}^*_{p,r}\right). \tag{4.7}$$

Inclusions (4.4) and (4.7) complete the proof. □

Theorem 4 means that every trajectory of the system can be approximated by a trajectory obtained by a full consumption of the control resource.

Theorem 4 implies the validity of the following corollary.

Corollary 3. *The equality*

$$\mathrm{cl}\,\left(\mathbf{Z}^*_{p,r}(t)\right) = \mathbf{Z}_{p,r}(t)$$

holds for every $t \in [a,b]$, *where* $\mathbf{Z}_{p,r}(t)$ *is defined by* (1.5) *and* $\mathbf{Z}^*_{p,r}(t) = \left\{x(t) \in \mathbb{R}^n \colon x(\cdot) \in \mathbf{Z}^*_{p,r}\right\}$.

Remark. Note that the results obtained in the paper are also valid for control systems described by similar types of ordinary differential equations and for systems described by Volterra type integral equations. The obtained results can be used in the modeling of control processes arising in physics, mechanics, biology, and economics, where control systems have a limited control resource such as energy, fuel, finance, food, etc. The continuity of the set of trajectories with respect to r and p implies that small errors in specifying r and p in the mathematical models will cause a small deviation of the set of trajectories. The robustness of the system's trajectory with respect to the remaining control resource means that an efficient way to achieve a desirable result is to consume the control resource in economy mode (in small quants) and to avoid an aggressive consumption of the available control resource.

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