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STABILITY REGION FOR DISCRETE TIME SYSTEMS AND ITS BOUNDARY

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In this paper we investigate the Schur stability region of the nth order polynomials in the coefficient space. Parametric description of the boundary set is obtained. We show that all the boundary can be obtained as a multilinear image of three $(n-1)$-dimensional boxes. For even and odd n these boundary boxes are different. Analogous properties for the classical multilinear reflection map are unknown. It is shown that for $n \geq 4$, both two parts of the boundary which are pieces of the corresponding hyperplanes are nonconvex. Polytopes in the nonconvex stability region are constructed. A number of examples are provided.

Keywords: Schur stability, stability region, polytope, boundary set.

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Исследуется область устойчивости по Шуру многочленов порядка n в пространстве коэффициентов. Получено параметрическое описание граничного множества. Показано, что вся граница может быть получена как мультилинейный образ трех $(n-1)$-мерных параллелепипедов, которые различны для четных и нечетных n. Аналогичные свойства для классического отображения отражения неизвестны. При $n \geq 4$ показана невыпуклость обеих частей границы, которые являются кусками соответствующих гиперплоскостей. Построены многогранники в невыпуклой области устойчивости. Приведено несколько примеров.

Ключевые слова: устойчивость по Шуру, область устойчивости, многогранник, граничное множество.

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1. Introduction

Consider the nth order real monic polynomial

$$a(s) = a_1 + a_2 s + \dots + a_n s^{n-1} + s^n, \tag{1.1}$$

which corresponds to the n-dimensional vector $a = (a_1, a_2, \dots, a_n)^T \in \mathbb{R}^n$. The polynomial (1.1) is called Schur (Hurwitz) stable if all its roots lie in the open unit disc (left half plane) of the complex plane.

For discrete time systems the polynomial (1.1) arises in the denominators of the transfer functions (or transfer matrix functions). The system is stable if and only if all poles lie in the open unit disc, that is, it is Schur stable. From now on, the term stable (H-stable) will mean Schur stable (Hurwitz stable).

Define

$$\mathcal{D}_n = \{a \in \mathbb{R}^n \colon \text{ Polynomial } a(s) \text{ is stable}\}.$$

The set $\mathcal{D}_2$ is an open triangle with vertices $(-1, 0)$, $(1, 2)$ and $(1, -2)$ in the (a_1, a_2)-plane. For $n \geq 3$ the set $\mathcal{D}_n$ is open, bounded, non-convex and the closed convex hull of $\mathcal{D}_n$ is a polytope with $(n+1)$ known vertices $V^1, V^2, \dots, V^{n+1}$, that is

$$\overline{\text{co}}\mathcal{D}_n = \text{co}\{V^1, V^2, \dots, V^{n+1}\},$$

where V^i are the coefficient vectors of the boundary polynomials $(s+1)^{n-i+1}(s-1)^{i-1}$, $(1 \leq i \leq n+1)$, $\overline{\text{co}}$ stands for the closure of the convex hull (see [1]). The volume of $\mathcal{D}_n$ has been calculated in [2].

In the case of $n = 2$ there exists a one to one and onto map $g : (-1, 1) \times (-1, 1) \to \mathcal{D}_2$, defined by $g(k_1, k_2) = (k_2, k_1k_2 + k_1)$, with $\mathcal{D}_2$ the open triangle with vertices $(-1, 0)$, $(1, 2)$ and $(1, -2)$. Based on this map defined for the case $n = 2$, we construct the following map $f : \mathbb{R}^n \to \mathbb{R}^n$ for an arbitrary $n \geq 3$ as follows:

If n is even then consider the following product of second order terms

$$\left[s^2 + (k_1k_2 + k_1)s + k_2\right] \cdot \left[s^2 + (k_3k_4 + k_3)s + k_4\right] \cdots \left[s^2 + (k_{n-1}k_n + k_{n-1})s + k_n\right]$$

(the total number of such terms is $n/2$). After multiplication we obtain the following nth order polynomial in the variable s:

$$f_1(k_1, \ldots, k_n) + f_2(k_1, \ldots, k_n)s + \cdots + f_n(k_1, \ldots, k_n)s^{n-1} + s^n.$$

The coefficient vector $(f_1(k_1, \ldots, k_n), f_2(k_1, \ldots, k_n), \ldots, f_n(k_1, \ldots, k_n))^T$ defines the required map $f \colon \mathbb{R}^n \to \mathbb{R}^n$.

If n is odd, then $n - 1$ is even and the above procedure defines the $(n - 1)$-th order polynomial

$$f_1(k_1, \ldots, k_{n-1}) + f_2(k_1, \ldots, k_{n-1})s + \cdots + f_{n-1}(k_1, \ldots, k_{n-1})s^{n-2} + s^{n-1},$$

and the multiplication of the last polynomial by $s + k_n$ defines the map $f : \mathbb{R}^n \to \mathbb{R}^n$ for odd n.

If n is even the map f is symmetrical with respect to the pairs (k_1, k_2), (k_3, k_4), …, (k_{n-1}, k_n). If n is odd it is symmetrical with respect to the pairs (k_1, k_2), (k_3, k_4), …, (k_{n-2}, k_{n-1}).

The map $f : \mathbb{R}^n \to \mathbb{R}^n$ defined above is multilinear, that is, affine linear with respect to each component. The set $Q = \{(x_1, \ldots, x_l)^T \in \mathbb{R}^l : \; a_1^- \leq x_1 \leq a_1^+, \ldots, a_l^- \leq x_l \leq a_l^+\}$ is called a box.

Theorem 1 (The Mapping Theorem [11]). *Suppose that Q is a box in $\mathbb{R}^l$ with the set of extreme points $\{q^i\}$ and $f : Q \to \mathbb{R}^k$ is multilinear. Then $\operatorname{co} f(Q) = \operatorname{co}\{f(q^i) \colon q^i$ is an extreme point of $Q\}$.*

Proposition 1. *The polynomial $a(s) = a_1 + a_2s + \cdots + a_ns^{n-1} + s^n$ is stable if and only if there exist numbers $k_j \in (-1, 1)$ such that $a_i = f_i(k_1, k_2, \ldots, k_n)$ $(i, j = 1, 2, \ldots n)$.*

Proof. The proof follows from the construction of f and g (see [4]). □

Define the cube $\mathcal{C} = \{(k_1, \ldots, k_n)^T : -1 \leq k_1 \leq 1, \ldots, -1 \leq k_n \leq 1\}$.

Let $\mathcal{C}^0$ and $\partial\mathcal{C}$ be the interior and the boundary sets of the set $\mathcal{C}$ respectively. For $i \in \{1, 2, \ldots, n\}$ define $(n - 1)$-dimensional faces $\{k_i = 1\} = \{k \in \mathcal{C} : k_i = 1\}$, $\{k_i = -1\} = \{k \in \mathcal{C} : k_i = -1\}$. Then

$$\mathcal{C}^0 = \{(k_1, k_2, \ldots, k_n)^T : -1 < k_1 < 1, \ldots, -1 < k_n < 1\},$$

$$\partial\mathcal{C} = \{k_1 = 1\} \cup \cdots \cup \{k_n = 1\} \cup \{k_1 = -1\} \cup \cdots \cup \{k_n = -1\}.$$

For any $i \in \{1, \ldots, n\}$, the sets $\{k_i = 1\}$ and $\{k_i = -1\}$ are $(n - 1)$-dimensional boxes. According to Proposition 1, $f(\mathcal{C}^0) = \mathcal{D}_n$.

Proposition 2. *$f(\partial\mathcal{C}) = \partial\mathcal{D}_n$, where $\partial\mathcal{D}_n$ is the boundary of the open set $\mathcal{D}_n$.*

Proof. The proof is based on Proposition 1. □

Another description of stable polynomials is the classical reflection map and the reflection coefficients (or Schur–Szego parameters). The reflection map is multilinear and can be defined recursively through matrix multiplications (see [3]).

In [3], using the classical reflection map and the reflection coefficients, the outside approximation of the stability region $\mathcal{D}_n$ is given by splitting the box $[-1, 1]^n = [-1, 1] \times \cdots \times [-1, 1]$ into subboxes. Convex approximation of the stability region by boxes are considered in [4; 5] and ellipsoidal approximation are considered in [6;7]. Topological and geometrical properties of the stability regions are studied in [8–10].

In this paper, by using f defined above, differing from the classical reflection map, we obtain parametric equations of the boundary surfaces of the stability region $\mathcal{D}_n$. It is shown that the boundary set $\partial\mathcal{D}_n$ can be obtained as multilinear images of three chosen faces of the cube $[-1,1]^n$ (Theorem 2 and 3), from which and from the Mapping Theorem it follows that their convex hull are polytopes. Note that the classical reflection map does not give similar properties of the stability region $\mathcal{D}_n$. We define polytopes in the nonconvex stability region (Theorem 6) and give two examples of application.

2. Description of the Boundary Set $\partial\mathcal{D}_n$

The boundary set $\partial\mathcal{D}_n$ corresponds to polynomials with at least one root $s = e^{j\theta}$, $\theta \in [0,\pi]$, ($j^2 = -1$):

$$a_1 + a_2 e^{j\theta} + a_3 e^{2j\theta} + \cdots + a_n e^{(n-1)j\theta} + e^{nj\theta} = 0$$

or

$$\begin{aligned} a_1 + a_2\cos\theta + \cdots + a_n\cos(n-1)\theta + \cos n\theta &= 0, \\ a_2\sin\theta + \cdots + a_n\sin(n-1)\theta + \sin n\theta &= 0. \end{aligned} \tag{2.1}$$

For $\theta = 0$ and $\theta = \pi$ the first equation in (2.1) give $(n-1)$-dimensional hyperplanes:

$$a_1 + a_2 + \cdots + a_n = -1, \tag{2.2}$$

$$a_1 - a_2 + \cdots + (-1)^{n-1} a_n = (-1)^{n+1}. \tag{2.3}$$

For $\theta \in (0,\pi)$ the system (2.1) gives an $(n-2)$-dimensional hyperplane as the intersection of two $(n-1)$-dimensional hyperplanes. Therefore, this part of the boundary is obtained by the movement of this $(n-2)$-dimensional hyperplane (see [1]).

Define the sets

$$\begin{aligned} B_1\ (B_{-1}) = \ & \big\{a \in \mathbb{R}^n : \text{ The polynomial } a(s) \text{ has all roots in the closed disc } |z| \le 1, \\ & \text{and has at least one root } s = 1\ (s = -1)\big\}, \\ B_c = \ & \big\{a \in \mathbb{R}^n : \text{ The polynomial } a(s) \text{ has all roots in the closed disc } |z| \le 1, \\ & \text{and has at least one complex root } s = e^{j\theta},\ 0 < \theta < \pi\big\}. \end{aligned}$$

Then

$$\partial\mathcal{D}_n = B_1 \cup B_{-1} \cup B_c.$$

As a result of the discussion above, B_1 (B_{-1}) is a piece of the hyperplane (2.2) ((2.3)) and B_c is a nonlinear n-dimensional surface. From the symmetry properties of f the following ensues

Proposition 3.

1) *If n is even then*

$$\begin{aligned} &f(\{k_1 = 1\}) = f(\{k_3 = 1\}) = \cdots = f(\{k_{n-1} = 1\}), \\ &f(\{k_1 = -1\}) = f(\{k_3 = -1\}) = \cdots = f(\{k_{n-1} = -1\}), \\ &f(\{k_2 = 1\}) = f(\{k_4 = 1\}) = \cdots = f(\{k_n = 1\}), \\ &f(\{k_2 = -1\}) = f(\{k_4 = -1\}) = \cdots = f(\{k_n = -1\}). \end{aligned}$$

2) *If n is odd then*

$$\begin{aligned} &f(\{k_1 = 1\}) = f(\{k_3 = 1\}) = \cdots = f(\{k_{n-2} = 1\}), \\ &f(\{k_1 = -1\}) = f(\{k_3 = -1\}) = \cdots = f(\{k_{n-2} = -1\}), \\ &f(\{k_2 = 1\}) = f(\{k_4 = 1\}) = \cdots = f(\{k_{n-1} = 1\}), \\ &f(\{k_2 = -1\}) = f(\{k_4 = -1\}) = \cdots = f(\{k_{n-1} = -1\}). \end{aligned}$$

Proposition 4. *Let $p(s) = a_1 + a_2 s + s^2$ be a second order polynomial. Then $p(s)$ has roots in the closed disc $\{z\colon |z| \le 1\}$ if and only if there exist real numbers k_1 and k_2 with $|k_1| \le 1$, $|k_2| \le 1$ such that*

$$p(s) = s^2 + (k_1 k_2 + k_1)s + k_2.$$

Proof. The proof of Proposition 4 follows from the properties of the map g defined in Section 1. □

Theorem 2. *Let n be even. Then*

$$1)\quad f(\{k_1 = 1\}) = B_{-1}, \qquad 2)\quad f(\{k_1 = -1\}) = B_1,$$
$$3)\quad f(\{k_2 = -1\}) \subset B_{-1} \cap B_1, \qquad 4)\quad f(\{k_2 = 1\}) = B_c.$$

Proof. 1) If $a \in f(\{k_1 = 1\})$ then $a(s)$ has a factor $s^2 + (k_1k_2 + k_1)s + k_2 = s^2 + (k_2 + 1)s + k_2 = (s+1)(s+k_2)$ and $a(s)$ has the root $s = -1$ and by Proposition 4 we have $a \in B_{-1}$.

Conversely, if $a \in B_{-1}$ then $a(s)$ has the factor $(s+1)$. Due to the fact that the number of complex roots is even and Proposition 4, $a(s)$ has a factor $(s+1)(s+k_2^0)$ and has the form

$$a(s) = (s+1)(s+k_2^0)[s^2 + (k_3^0 k_4^0 + k_3^0)s + k_4^0] \cdots [s^2 + (k_{n-1}^0 k_n^0 + k_{n-1}^0)s + k_n^0]$$

with $|k_i^0| \le 1$ $(i = 2, 3, \ldots, n)$, which gives the equality $a = f(1, k_1^0, \ldots, k_n^0)$. Therefore

$$a \in f(\{k_1 = 1\}).$$

2) The proof is similar.

3) If $a \in f(\{k_1 = -1\})$ then the corresponding polynomial $a(s)$ has a factor

$$s^2 + (k_1k_2 + k_1)s + k_2 = s^2 + (k_1 - k_1)s - 1 = (s-1)(s+1).$$

Therefore, it follows from Proposition 4 that $a \in B_{-1} \cap B_1$.

4) If $a \in f(\{k_2 = 1\})$ then $a(s)$ has a factor

$$s^2 + (k_1k_2 + k_1)s + k_2 = s^2 + 2k_1 s + 1 = (s+k_1)^2 + (1 - k_1^2)$$

and has complex roots if $|k_1| < 1$ (for $k_1 = 1$ or $k_1 = -1$ see 1) and 2)).

Conversely, if $a \in B_c$, then $a(s)$ has complex roots $s = e^{\pm j\theta}$ $(0 < \theta < \pi,\ j^2 = -1)$ and a second order factor

$$(s - e^{j\theta})(s - e^{-j\theta}) = s^2 - 2\cos\theta \cdot s + 1 = s^2 + (k_1 \cdot 1 + k_1) + 1,$$

where $k_1 = -\cos\theta$. Therefore $a \in f(\{k_2 = 1\})$. (We have implicitly used Proposition 4.)

Proposition 5. *Let n be odd. Then*

$$1)\quad f(\{k_1 = 1\}) \subset f(\{k_n = 1\}), \quad 2)\quad f(\{k_1 = -1\}) \subset f(\{k_n = -1\}).$$

Proof. 1) If $k_1 = 1$ then the factorization which defines the map f is as follows (see Section 1)

$$[s^2 + (k_2+1)s + k_2] \cdots [s^2 + (k_{n-2}k_{n-1} + k_{n-2})s + k_{n-1}](s + k_n). \tag{2.4}$$

Similarly, if $k_n = 1$ the factorization is

$$[s^2 + (k_1k_2 + k_1)s + k_2] \cdots [s^2 + (k_{n-2}k_{n-1} + k_{n-2})s + k_{n-1}](s+1). \tag{2.5}$$

Consider the multiplication of the first and the last terms in both (2.4) and (2.5) (the remaining factors are the same)

$$[s^2 + (k_2+1)s + k_2](s + k_n) = (s+k_2)(s+k_n)(s+1), \tag{2.6}$$

$$[s^2 + (k_1k_2 + k_1)s + k_2](s + 1). \tag{2.7}$$

Comparing (2.6) and (2.7) for $-1 \le k_1 \le 1$, $-1 \le k_2 \le 1$ and $-1 \le k_n \le 1$, we may see that the factor $(s + k_2)(s + k_n)$ gives only real roots in (2.6), whereas the factor $s^2 + (k_1k_2 + k_1)s + k_2$ in (2.7) gives all roots from the closed disc $|z| \le 1$ according to [1].

Therefore $f(\{k_1 = 1\}) \subset f(\{k_n = 1\})$.

2) The proof is similar. □

Theorem 3. *Let n be odd. Then*

1) $f(\{k_n = 1\}) = B_{-1}$, 2) $f(\{k_n = -1\}) = B_1$,

3) $f(\{k_2 = -1\}) \subset B_{-1} \cap B_1$, 4) $f(\{k_2 = 1\}) = B_c$.

Proof. We prove only 1); the proofs of 2)–4) can be carried out by analogy.

If $a \in f(\{k_n = 1\})$ then $a(s)$ has the factor $(s + 1)$ (see the definition of f), therefore, in view of Proposition 4, $a \in B_{-1}$.

Conversely, if $a \in B_{-1}$, then $a(s) = b(s) \cdot (s + 1)$ where $b(s)$ is an even order polynomial having all roots in the disc $|z| \le 1$. The polynomial $b(s)$ has the factorization

$$b(s) = \left[s^2 + (k_1k_2 + k_1)s + k_2\right] \cdots \left[s^2 + (k_{n-2}k_{n-1} + k_{n-2})s + k_{n-1}\right]$$

and, by Proposition 4, $|k_i| \le 1$ for all $i = 1, 2, \ldots, n - 1$. Therefore

$$a(s) = \left[s^2 + (k_1k_2 + k_1)s + k_2\right] \cdots \left[s^2 + (k_{n-2}k_{n-1} + k_{n-2})s + k_{n-1}\right](s + 1)$$

and $a \in f\{k_n = 1\}$. □

The above results show that if n is even then:

i) The images of the faces $\{k_1 = 1\}$, $\{k_3 = 1\}, \ldots, \{k_{n-1} = 1\}$ under f are the same and equal to B_{-1}.

ii) The images of the faces $\{k_1 = -1\}$, $\{k_3 = -1\}, \ldots, \{k_{n-1} = -1\}$ are the same and equal to B_1.

iii) The images of the faces $\{k_2 = 1\}$, $\{k_4 = 1\}, \ldots, \{k_n = 1\}$ are the same and equal to B_c.

iv) The images of the faces $\{k_2 = -1\}$, $\{k_4 = -1\}, \ldots, \{k_n = -1\}$ are the same and are contained in $B_{-1} \cap B_1$.

If n is odd then:

i) The image of the face $\{k_n = 1\}$ equals to B_{-1}. The image of the face $\{k_n = -1\}$ equals to B_1.

ii) The images of the faces $\{k_1 = 1\}$, $\{k_3 = 1\}, \ldots, \{k_{n-2} = 1\}$ are the same and are contained in B_{-1}.

iii) The images of the faces $\{k_1 = -1\}$, $\{k_3 = -1\}, \ldots, \{k_{n-2} = -1\}$ are the same and are contained in B_1.

iv) The images of the faces $\{k_2 = 1\}$, $\{k_4 = 1\}, \ldots, \{k_{n-1} = 1\}$ are the same and equal to B_c.

v) The images of the faces $\{k_2 = -1\}$, $\{k_4 = -1\}, \ldots, \{k_{n-1} = -1\}$ are the same and are contained in $B_{-1} \cap B_1$.

Corollary 1. *Assume that n is even. Then the surface B_1 (B_{-1}) has the parametric equation*

$$x_i = f_i(-1, k_2, \ldots, k_n), \quad (x_i = f_i(1, k_2, \ldots, k_n))$$
$$-1 \le k_2 \le 1, \ldots, \; -1 \le k_n \le 1, \quad i = 1, 2, \ldots, n.$$

Corollary 2. *Assume that n is odd. Then the surface B_1 (B_{-1}) has the parametric equation*

$$x_i = f_i(k_1, \ldots, k_{n-1}, -1), \quad (x_i = f_i(k_1, \ldots, k_{n-1}, 1))$$
$$-1 \le k_1 \le 1, \ldots, \; -1 \le k_{n-1} \le 1, \quad i = 1, 2, \ldots, n.$$

Corollary 3. *The surface $B_{\mathcal{C}}$ has the parametric equation*

$$x_i = f_i(k_1, 1, k_3, \dots, k_n), \quad (i = 1, 2, \dots, n)$$
$$-1 \le k_1 \le 1,\ -1 \le k_3 \le 1, \dots,\ -1 \le k_n \le 1.$$

Theorem 4. *The equalities*

$$\mathrm{co}B_{-1} = \mathrm{co}\left\{V^1, V^2, \dots, V^n\right\}, \tag{2.8}$$

$$\mathrm{co}B_1 = \mathrm{co}\left\{V^2, V^3, \dots, V^{n+1}\right\} \tag{2.9}$$

are satisfied.

Proof of (2.8). Without loss of generality assume that n is even. Employing Theorem 2, $B_{-1} = f(\{k_1 = 1\})$ we have

$$\mathrm{co}B_{-1} = \mathrm{co}f(\{k_1 = 1\}). \tag{2.10}$$

On using Theorem 1, it follows that

$$\mathrm{co}f(\{k_1 = 1\}) = \mathrm{co}\{f(k^1), f(k^2), \dots, f(k^m)\}, \tag{2.11}$$

where k^j $(j = 1, 2, \dots, m)$ are the extreme points of the $(n-1)$-dimensional box $\{k_1 = 1\}$, that is $k_1 = 1$, $k_2 = \pm 1$, …, $k_n = \pm 1$. Each $f(k^j)$ are the coefficient vector obtained after multiplication of second order factors.

The first factor $s^2 + (k_1 k_2 + k_1)s + k_2$ with $k_1 = 1$, $k_2 = \pm 1$ gives two polynomials $(s+1)^2$ and $(s^2 - 1)$, whereas the remaining factors $s^2 + (k_{t-1}k_t + k_{t-1})s + k_t$, $(4 \le t \le n)$ with $k_{t-1} = \pm 1$, $k_t = \pm 1$ give three polynomials $(s+1)^2$, $(s-1)^2$ and $(s^2 - 1)$. Therefore, the extreme points of $\{k_1 = 1\}$ give n polynomials

$$(s+1)^n,\ (s+1)^{n-1}(s-1), \dots, (s+1)(s-1)^{n-1}.$$

In summary, from the latter and equalities (2.10), (2.11), the equality (2.8) follows.

The proof of (2.9) is similar and is omitted. □

Theorem 5. 1) *For $n = 3$,*

$$B_{-1} = \mathrm{co}\{V^1, V^2, V^3\}, \tag{2.12}$$

$$B_1 = \mathrm{co}\{V^2, V^3, V^4\}. \tag{2.13}$$

2) *For $n \ge 4$, the sets $\mathrm{co}\{V^1, \dots, V^n\}$ and $\mathrm{co}\{V^2, \dots, V^{n+1}\}$ contain nonboundary (exterior to $\mathcal{D}_n$) points.*

Proof. Recall that $V^i \in \mathbb{R}^3$ and correspond to the polynomials $(s+1)^{4-i}(s-1)^{i-1}$ $(i = 1, 2, 3, 4)$.

1) We prove (2.12) (The proof of (2.13) is similar). It follows from Theorem 4 that

$$B_{-1} \subset \mathrm{co}\{V^1, V^2, V^3\}. \tag{2.14}$$

Consider the segments $[V^1, V^2]$, $[V^2, V^3]$ and $[V^1, V^3]$. The inclusions $[V^1, V^2] \subset B_{-1}$ and $[V^2, V^3] \subset B_{-1}$ are straightforward. Let us prove $[V^1, V^3] \subset B_{-1}$. Set $\lambda \in [0, 1]$ and consider $\lambda V^1 + (1-\lambda)V^3$ and the corresponding polynomial

$$\lambda(s+1)^3 + (1-\lambda)(s+1)(s-1)^2 = (s+1)\left[s^2 + (4\lambda - 2)s + 1\right].$$

This polynomial has the root $s = -1$ and its second order factor $s^2 + (4\lambda - 2)s + 1$ has roots

$$s = 1 - 2\lambda \pm 2\sqrt{\lambda - \lambda^2}\, j.$$

These roots have module 1:

$$|s|^2 = (1-2\lambda)^2 + 4(\lambda - \lambda^2) = 1.$$

By the definition of B_{-1}, we have $[V^1, V^3] \subset B_{-1}$. As a consequence of this fact and (2.14) the equality (2.12) is satisfied.

2) Consider the segment

$$\begin{aligned} [V^1, V^4] &= [(s+1)^n, (s+1)^{n-3}(s-1)^3] \\ &= \{\lambda(s+1)^n + (1-\lambda)(s+1)^{n-3}(s-1)^3 : \ \lambda \in [0,1]\}. \end{aligned}$$

For $\lambda = \dfrac{1}{2}$, the corresponding polynomial has the factor $\frac{1}{2}(s+1)^3 + \frac{1}{2}(s-1)^3 = s^3 + 3s$ with the nonboundary (exterior) roots $s = \pm\sqrt{3}j$. This shows that $\mathrm{co}\{V^1, \dots, V^n\}$ contains nonboundary points.

The same is true for $\mathrm{co}\{V^2, \dots, V^{n+1}\}$. □

Example 1. Consider robust stability problem for the following multilinear family

$$\begin{aligned} a(s,q) &= s^6 + (2.25 + 1.2q_1)s^5 + (-0.25 - 0.9q_1 + 0.8q_2)s^4 \\ &\quad + (3.375 + 1.8q_1 + 1.8q_2 + 0.96q_1q_2)s^3 - (1 + 0.9q_1 + q_2 + 0.72q_1q_2)s^2 \\ &\quad + (1.125 + 0.6q_1 + 0.9q_2 + 0.48q_1q_2)s + 0.2q_2 + 0.25, \end{aligned}$$

$q_1 \in [-3.6, -1.5]$, $q_2 \in [-2.2, -0.5]$. For $q_1 = -2.55$, $q_2 = -1.35$ the polynomial is stable. Using the equations of the boundary $\partial\mathcal{D}_n$ in the parametric forms (Corollaries 1–3) write three multilinear systems of equations

$$f_i(1, k_2, k_3, \dots, k_6) - a_i(q_1, q_2) = 0, \quad (i = 1, \dots, 6) \tag{2.15}$$

$$f_i(-1, k_2, k_3, \dots, k_6) - a_i(q_1, q_2) = 0, \quad (i = 1, \dots, 6) \tag{2.16}$$

$$f_i(k_1, 1, k_3, \dots, k_6) - a_i(q_1, q_2) = 0, \quad (i = 1, \dots, 6) \tag{2.17}$$

where $a_i(q_1, q_2)$ are the coefficients of $a(s,q)$ and

$$(k_1, \dots, k_6, q_1, q_2) \in B = [-1,1] \times \cdots \times [-1,1] \times [-3.6, -1.5] \times [-2.2, -0.5].$$

We have to show that all three systems (2.15)–(2.17) have no solutions. Here we use splitting-elimination algorithm (see [4]) with the use of The Mapping Theorem (divide the box B into small subboxes, if for a subbox the zero is not contained in the convex hull of the images of vertices, then eliminate this subbox).

For the systems (2.15), (2.16) all subboxes are eliminated after 2 steps in 0.02 sec. For the system (2.17) all subboxes are eliminated after 4240 steps in 118 sec. Therefore, the given family does not intersect the boundary of $\mathcal{D}_n$ and has a stable member. Consequently, the family is robust stable. □

Note that this multilinear family can be extended to an affine family with 4 extreme polynomials corresponding to the extreme point of (q_1, q_2). However, this extension is not stable since the segment $[b(z), c(z)]$ is not stable, where $b(z)$ corresponds to $q_1 = -3.6$, $q_2 = -0.5$ and $c(z)$ corresponds to $q_1 = -1.5$, $q_2 = -2.2$. Note also that if we use the classical reflection map then the number of equations like (2.15)–(2.17) increases from 3 to 12.

Example 2. Consider for a stable member in the family

$$\begin{aligned} a(s,q) &= s^6 + (-q_1q_2 + 2q_1 - 5q_2 + 8)s^5 + (q_1q_2 - 2q_1 + 6q_2 - 11)s^4 \\ &\quad + (2q_1q_2 - 3q_1 + 7q_2 - 12)s^3 + (-q_1q_2 - q_1 - 2q_2 - 8)s^2 \\ &\quad + (2q_1q_2 - q_1 + 7q_2 - 1)s - 2q_1 + 2q_2 - 14, \end{aligned}$$

$q_1 \in [-1, 2]$, $q_2 \in [-1, 2]$. For $q_1 = q_2 = 0$ the obtained polynomial is unstable. Three systems of multilinear equations like (2.15)–(2.17) have no solutions on the box $B = [-1,1] \times \cdots \times [-1,1] \times [-1,2] \times [-1,2]$. Splitting-elimination algorithm gives the following answers:

For systems like (2.15), (2.16), and (2.17) all subboxes are eliminated after 2 steps and there are no solutions. Consequently, the family has no stable member. □

3. Stable Polytopes

In this section, using the results from [12] and the Mobius transformation, we construct stable polytopes. Given the nth order polynomial $p(z)$ with positive coefficients its Mobius transformation polynomial is defined by $\tilde{p}(s) = \mathcal{M}\{p(z)\} = (s-1)^n p\Big(\frac{s+1}{s-1}\Big)$. It is well known that $p(z)$ is stable if and only if $\tilde{p}(s)$ is H-stable.

As in [12], we use second order factors in the construction of stable polytopes. We observe that the results in [12] are stated for the monic case, however, these results can easily be transformed to the non-monic case. In [12], for a second order monic factor $s^2 + \alpha\, s + \alpha$ the condition $\alpha \geq 1$ is required; in the non-monic case any second order factor $a\, s^2 + \alpha\, s + \alpha$ must satisfy the conditions $a > 0$, $\alpha \geq a$.

Firstly assume that n is even, $n = 2m$ and $A = \{(x, y) \in \mathbb{R}^2 : \ y < 0, y > -x\}$, $(x_i, y_i) \in A$ $(i = 1, 2, \dots, m)$. Define the nth order $(n+1)$ polynomials

$$\begin{aligned}
p_0(z) &= \Big(x_1 z^2 + y_1 z + \frac{x_1+y_1}{3}\Big)\Big(x_2 z^2 + y_2 z + \frac{x_2+y_2}{3}\Big)\cdots\Big(x_m z^2 + y_m z + \frac{x_m+y_m}{2}\Big),\\
p_1(z) &= \Big(\frac{4x_1+y_1}{6} z^2 + y_1 z + \frac{4x_1+y_1}{6}\Big)\Big(x_2 z^2 + y_2 z + \frac{x_2+y_2}{3}\Big)\cdots\\
&\qquad \cdots\Big(x_m z^2 + y_m z + \frac{x_m+y_m}{2}\Big),\\
p_2(z) &= \Big(\frac{4x_1+y_1}{6} z^2 + \frac{2(x_1+y_1)}{3} z + \frac{y_1}{2}\Big)\Big(x_2 z^2 + y_2 z + \frac{x_2+y_2}{3}\Big)\cdots\\
&\qquad \cdots\Big(x_m z^2 + y_m z + \frac{x_m+y_m}{2}\Big),\\
p_3(z) &= \Big(x_1 z^2 + y_1 z + \frac{x_1+y_1}{3}\Big)\Big(\frac{4x_2+y_2}{6} z^2 + y_2 z + \frac{4x_2+y_2}{6}\Big)\cdots\\
&\qquad \cdots\Big(x_m z^2 + y_m z + \frac{x_m+y_m}{2}\Big),\\
&\ \ \vdots\\
p_n(z) &= \Big(x_1 z^2 + y_1 z + \frac{x_1+y_1}{3}\Big)\Big(x_2 z^2 + y_2 z + \frac{x_2+y_2}{3}\Big)\cdots\\
&\qquad \cdots\Big(\frac{4x_m+y_m}{6} z^2 + \frac{2(x_m+y_m)}{3} z + \frac{y_m}{2}\Big).
\end{aligned}$$

If n is odd, $n = 2m-1$ then

$$\begin{aligned}
p_0(z) &= \Big(x_1 z^2 + y_1 z + \frac{x_1+y_1}{3}\Big)\cdots\Big(x_{m-1} z^2 + y_{m-1} z + \frac{x_{m-1}+y_{m-1}}{3}\Big)\Big(x_m z + y_m\Big),\\
p_1(z) &= \Big(\frac{4x_1+y_1}{6} z^2 + y_1 z + \frac{4x_1+y_1}{6}\Big)\cdots\Big(x_{m-1} z^2 + y_{m-1} z + \frac{x_{m-1}+y_{m-1}}{3}\Big)\Big(x_m z + y_m\Big),\\
&\ \ \vdots\\
p_{n-1}(z) &= \Big(x_1 z^2 + y_1 z + \frac{x_1+y_1}{3}\Big)\cdots\Big(\frac{4x_{m-1}+y_{m-1}}{6} z^2 + \frac{2(x_{m-1}+y_{m-1})}{3} z + \frac{y_{m-1}}{2}\Big)\\
&\qquad \times\Big(x_m z + y_m\Big),\\
p_n(z) &= \Big(x_1 z^2 + y_1 z + \frac{x_1+y_1}{3}\Big)\cdots\Big(x_{m-1} z^2 + y_{m-1} z + \frac{x_{m-1}+y_{m-1}}{2}\Big)\\
&\qquad \times\Big(\frac{x_m+y_m}{2} z + \frac{x_m+y_m}{2}\Big).
\end{aligned}$$

Define the polytope

$$\mathcal{P} = \mathrm{co}\{p_0(z), p_1(z), \dots, p_n(z)\}. \tag{3.1}$$

Theorem 6. *Assume that $(x_i, y_i) \in A$ $(i = 1, 2, \dots, m)$, where $n = 2m$ if n is even and $n = 2m-1$ if n is odd. Then the inner points of the polytope $\mathcal{P}$ defined by* (3.1) *are stable.*

Proof. We observe that if $p(z) = p_1(z) \cdot p_2(z)$ then $\mathcal{M}\{p(z)\} = \mathcal{M}\{p_1(z)\} \cdot \mathcal{M}\{p_2(z)\}$ and for the two nth order polynomials $p(z)$ and $q(z)$ with positive coefficients $\mathcal{M}\{[p(z), q(z)]\} =$

$[\mathcal{M}\{p(z)\}, \mathcal{M}\{q(z)\}]$. Consequently, the segment $[p(z), q(z)]$ is stable if and only if the segment $[\tilde{p}(z), \tilde{q}(z)] = [\mathcal{M}\{p(z)\}, \mathcal{M}\{q(z)\}]$ is H-stable. By the Edge Theorem the polytope $\mathcal{P}$ is stable if all edges $[p_i(z), p_j(z)]$ are stable. Additionally

$$\begin{aligned}
\mathcal{M}\Big\{xz^2 + yz + \frac{x+y}{3}\Big\} &= \frac{4(x+y)}{3}s^2 + \frac{4x-2y}{3}s + \frac{4x-2y}{3},\\
\mathcal{M}\Big\{\frac{4x+y}{6}z^2 + yz + \frac{4x+y}{6}\Big\} &= \frac{4(x+y)}{3}s^2 + \frac{4x-2y}{3},\\
\mathcal{M}\Big\{\frac{4x+y}{6}z^2 + \frac{2(x+y)}{3}z + \frac{y}{2}\Big\} &= \frac{4(x+y)}{3}s^2 + \frac{4x-2y}{3}s,\\
\mathcal{M}\Big\{xz+y\Big\} &= (x+y)s + (x-y), \quad \mathcal{M}\Big\{\frac{x+y}{2}z + \frac{x+y}{2}\Big\} = (x+y)s.
\end{aligned}$$

Summarizing, the second order factors and the first order factor (in the case of odd n) of the Mobius transformation polynomials satisfy the required conditions of Theorems 2 and 3 from [12]. All edges of $\mathcal{P}$ are, therefore, either stable or lie on the stability boundary and consequently the inner points of $\mathcal{P}$ are stable.

Example 3 (Stabilization). Consider the stabilization problem for the transfer function $G(z) = \dfrac{z+1}{42z^3 - 47z^2 - 50z - 9}$ with the controller $C(z) = \dfrac{c_1 z + c_2}{z^2 + c_3}$. The closed loop characteristic polynomial is $p(z, c) = 42z^5 - 47z^4 + (-50 + 42c_3)z^3 + (-9 + c_1 - 47c_3)z^2 + (c_1 + c_2 - 50c_3)z + c_2 - 9c_3$. This family defines an affine subset $\mathcal{A}$ in the coefficient space and the problem consists of determining the values of c_1, c_2, and c_3 for which $p(z, c)$ is stable. Choose $(x_1, y_1) = (5, -2)$, $(x_2, y_2) = (7, -4)$, $(x_3, y_3) = (2, -1)$ (see Theorem 6). These values define the polytope $\mathcal{P}$ (see (3.1)) and the intersection problem of $\mathcal{P}$ and $\mathcal{A}$ is a standart linear programming problem (see [4]). Solving them gives a nonempty intersection and the projection of this intersection to the space (c_1, c_2, c_3) gives the following polytope of stablizing parameters (c_1, c_2, c_3): $\mathrm{co}\{(84.136, 16.532, 1.920),\ (85.231, 17.405, 1.972),\ (88.527, 17.837, 2.039),\ (85.553, 16.070, 1.929),\ (87.041, 17.081, 1.994),\ (89.510, 17.748, 2.053)\}$ (see Fig. 1(a)). □

Example 4 (Stabilization). Consider a similar problem as in Example 3 with the transfer function $G(z) = \dfrac{z+1}{23z^2 - 17z - 10}$ and controller $C(z) = \dfrac{c_1 z + c_2}{z^2 + c_3}$. Choose (x_1, y_1) and (x_2, y_2) as in Example 3. The polytope of stablizing parameters (c_1, c_2, c_3) is shown in Fig. 1(b). □

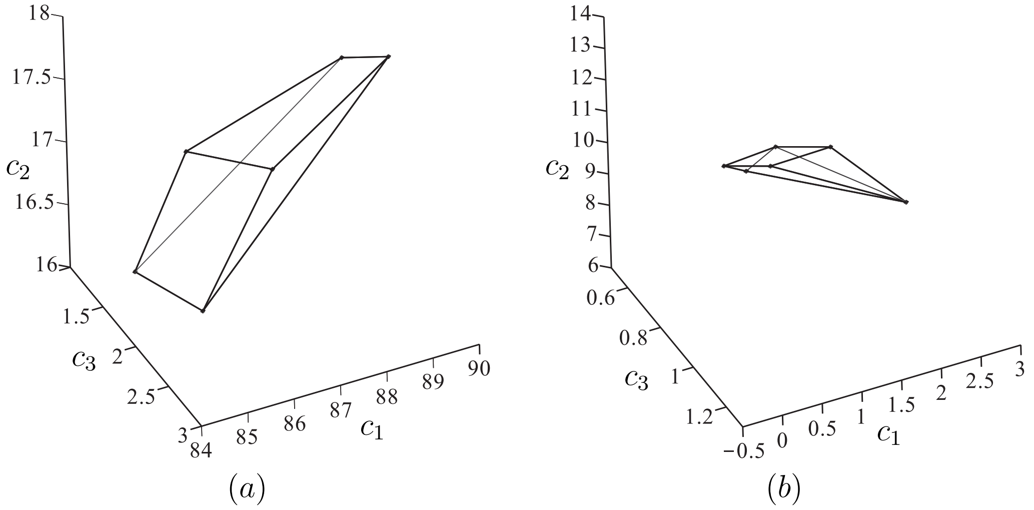

Fig. 1. Polytopes of stabilizing c for Example 3 and Example 4.

Results

The stability region of discrete time systems in the coefficient space is investigated. It is shown that the boundary of this region consists of three parts. The first and the second parts are nonconvex subsets of the corresponding hyperplanes, the third part is a nonlinear surface. The parametric equations of all three parts are obtained. Polytopes in the nonconvex stability region are defined. The obtained results can be applied in the robust stability and instability problems of a given multilinear family, in the stabilization and other related problems.

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