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A SURVEY OF HOPF–LAX FORMULAS AND QUASICONVEXITY IN PDES

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This is a short survey of recent results obtained by the author and collaborators primarily on Hopf–Lax formulas for Hamilton–Jacobi equations and obstacle problems. The initiation of the use of quasiconvex (i.e., level convex) functions in L^∞ control and differential games led to such formulas and is briefly reviewed. Dedicated to the memory of Academician A. I. Subbotin.

Keywords: Hopf–Lax, viscosity solution, Hamilton–Jacobi, quasiconvex.

Е. Н. Баррон. Обзор формул Хопфа — Лакса и квазивыпуклость дифференциальных уравнений в частных производных.

Статья представляет собой краткий обзор результатов, полученных автором и коллегами и касающихся в основном формул Хопфа — Лакса для уравнений Гамильтона — Якоби и задач с препятствием. К использованию таких формул привело начало применения квазивыпуклых функций (т. е. функций с выпуклыми множествами уровня) для управления в L^∞ и дифференциальных игр, что также рассматривается в обзоре. Посвящается памяти академика А.И. Субботина.

Ключевые слова: Хопф, Лакс, вязкостное решение, Гамильтон, Якоби, квазивыпуклость.

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Introduction

Many important problems in the calculus of variations, optimal control, and differential games lead to Hamilton–Jacobi equations in which the Hamiltonian is spatially independent. In such cases it was shown since the 1950s by P. Lax, E. Hopf, and O. Oleinik, that it is possible to derive an explicit solution for such Hamilton–Jacobi equations in the presence of some form of convexity. When the theory of viscosity solutions and minimax solutions of first and second order pdes was introduced and developed in the 1980s, attention returned to expanding the basis for such explicit solutions to more general conditions. All such formulas for these problems are now commonly referred to as Hopf–Lax, or Hopf–Lax–Oleinik, formulas.

In the 1990s explicit solutions were also obtained for equations permitting dependence on the solution in the Hamiltonian. Convexity plays an important, if not critical role in the derivation of explicit solutions and since the 1990s the theory of quasiconvex functions has also entered the scene. Original formulas were derived using optimal control and more recent formulas were derived using optimal control and variational calculus in L^∞. In 1995 A. Subbotin was able to extend the classical Hopf formula to first order obstacle problems. This survey describes the results obtained by the author and collaborators regarding both the basic problems, the obstacle problems, and the associated quasiconvex duality theory developed and used in the extended Hopf–Lax–Oleinik formulas.

1. Classical Hopf–Lax Formulas

Consider the Hamilton–Jacobi equation

$$u_t + H(Du) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n, \quad u(T,x) = g(x), \quad x \in \mathbb{R}^n. \tag{1.1}$$

1. We begin with the classical Lax or Lax–Oleinik formula which applies when H is convex and g is Lipschitz. These equations arise in consideration of the classical calculus of variations or optimal control problems in which the cost functionals do not have spatial dependence.

The unique viscosity solution of (1.1) is

$$u(t,x) = \sup_{y \in \mathbb{R}^n} g(y) - (T-t)H^*\left(\frac{y-x}{T-t}\right), \tag{1.2}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is assumed convex, continuous and coercive, i.e., $H(p)/|p| \to \infty$ as $|p| \to \infty$. Here $H^*(y) = \sup_{p \in \mathbb{R}^n} (p \cdot y - H(p))$ is the Fenchel conjugate of H.

2. If g is assumed convex and H is Lipschitz we have the Hopf formula in which the unique viscosity solution of (1.1) is

$$u(t,x) = (g^* - (T-t)H(\cdot))^*(x) = \sup_{y \in \mathbb{R}^n} y \cdot x - g^*(y) + (T-t)H(y),$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is assumed convex. This formula may either be proved directly to provide a solution or, as Bardi and Evans showed in [3] by the use of an associated differential game.

These formulas have been extended to the case when the terminal data g is merely lower semicontinuous and possibly infinite. Refer to [2; 13; 14] and [1] for further details and proofs of these results.

1.1. What Happens when $u_t + H(u, Du) = 0$?

To answer this question we consider the method of characteristics. The equations for the characteristics of the Hamilton–Jacobi equation with u dependence are

$$\dot{\xi} = H_p(\eta, p), \quad \dot{\eta} = -H(\eta, p) + p \cdot H_p(\eta, p), \quad \dot{p} = -H_u(\eta, p)p.$$

Based on the classical formulas, we would like the $\xi-$solution of the characteristics to be straight lines, which means we need $H_p(\eta(t), p(t)) = C$ for all $t > 0$. Differentiating this with respect to t we get the sufficient condition for straight line trajectories to be

$$0 = H_{up}(-H + p \cdot h_p) - H_{pp} \cdot pH_u. \tag{1.3}$$

If $H_u = 0$, it is obvious that (1.3) is satisfied. This leads eventually to the classical formulas of Lax and Hopf when the Hamiltonian is independent of u.

Now consider if we impose the condition that $p \mapsto H(\gamma, p)$ is homogeneous degree one, i.e., $H(\gamma, \lambda p) = \lambda H(\gamma, p)$. Differentiating with respect to λ and setting $\lambda = 1$ gives $-H + pH_p = 0$. The homogeneity assumption gets rid of the first term in (1.3). Differentiating the homogeneity assumption equation twice with respect to λ and setting $\lambda = 1$ gives $\langle p \cdot H_{pp}, p \rangle = 0$. If we also assume that $p \mapsto H(\gamma, p)$ is convex, H_{pp} is a nonnegative definite matrix and we conclude that, under this additional assumption $H_{pp} = 0$. Consequently, under the two conditions $p \mapsto H(\gamma, p)$ is homogeneous degree one and convex, we get (1.3) satisfied.

This leads to the conjecture that if H_u has a sign and $H(\gamma, p)$ is convex and homogeneous degree 1 in p a Lax and Hopf formula are possible. These formulas are discussed next.

1.2. Hopf–Lax Formulas with u Dependence

The derivation of the classical Lax formula uses optimal control theory and involves the use of Fenchel conjugates of convex functions. In this spirit we begin with optimal control in L^∞ leading to the consideration of quasiconvex functions and quasiconvex conjugates.[1] The simplest L^∞ control problem is

$$\begin{aligned}&\dot{\xi} = f(\tau,\xi,\zeta), \quad t<\tau\le T, \quad \xi(t)=x\in\mathbb{R}^n,\\ &P(\zeta) = g(\xi(T)) \vee \operatorname*{ess\,sup}_{\tau\in[t,T]} h(\tau,\xi(\tau),\zeta(\tau)),\\ &V(t,x) = \inf_{\zeta\in\mathcal{Z}[t,T]} P(\zeta).\end{aligned}$$

The value function $V:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ is the unique viscosity solution of

$$V_t + H(t,x,V,D_xV) = 0, \quad (t,x)\in[0,T]\times\mathbb{R}^n, \quad V(T,x) = g(x)\vee \min_z h(T,x,z), \tag{1.4}$$

where

$$H(t,x,r,p) = \begin{cases}\min_{z\in\mathcal{A}[t,x,r]} p\cdot f(t,x,z), & \mathcal{A}[t,x,r]\neq\varnothing,\\ +\infty, & \mathcal{A}[t,x,r]=\varnothing\end{cases}$$

and $\mathcal{A}[t,x,r] = \{z\in Z \mid h(t,x,z)\le r\}$.

Notice the occurrence of V dependence in (1.4) in which the Hamiltonian is nonincreasing in V. Furthermore, the Hamiltonian is positively homogeneous degree one in p. This leads us to suspect that a Lax formula is obtainable.

Toward this end, consider the equation with an initial value, rather than a terminal value:

$$u_t + H(u,Du) = 0, \quad (t,x)\in[0,\infty)\times\mathbb{R}^n, \quad u(0,x)=g(x), \quad x\in\mathbb{R}^n. \tag{1.5}$$

It is assumed that $\gamma\mapsto H(\gamma,p)$ is nondecreasing for each $p\in\mathbb{R}^n$ and $p\mapsto H(\gamma,p)$ is positively homogeneous degree one and convex.

Define

$$H^{\#}(z) = \inf\big\{\gamma \mid \sup_{p\in\mathbb{R}^n} p\cdot z - H(\gamma,p)\le 0\big\}.$$

It can be shown that $H^{\#}(z)$ is a quasiconvex function and we can represent the Hamiltonian $H(\gamma,p)$ as

$$H(\gamma,p) = \sup\{p\cdot z \mid z\in A[\gamma]\}, \quad A[\gamma] = \{z\in\mathbb{R}^n \mid H^{\#}(z)\le\gamma\}.$$

With this representation we can write the solution of (1.5) as the value function of an L^∞ control problem:

$$u(t,x) = \inf_\zeta g(\xi(t))\vee \operatorname*{ess\,sup}_{\tau\in[0,t]} H^{\#}(\zeta(\tau)), \quad \dot{\xi}(\tau)=\zeta(\tau), \quad 0<\tau\le t, \quad \xi(0)=x.$$

Choosing $\zeta(\tau)\equiv z\in\mathbb{R}^n$ we get

$$u(t,x)\le g(x+z\,t)\vee H^{\#}(z) \implies u(t,x)\le \inf_z g(x+z\,t)\vee H^{\#}(z).$$

For the other side we need an extended Jensen inequality for quasiconvex functions discovered in [10].

Theorem 1. *If $f:\mathbb{R}^n\to\mathbb{R}$ is quasiconvex and h is any integrable function, then*

$$f\Big(\int_\Omega h(x)\,d\mu(x)\Big) \le \mu - \operatorname*{ess\,sup}_{x\in\Omega} f(h(x)),$$

for any probability measure μ.

[1]Recall that a function $f:\mathbb{R}^n\to\mathbb{R}$ is quasiconvex if the level set $E_\alpha = \{x\in\mathbb{R}^n \mid f(x)\le\alpha\}$ is convex for every $\alpha\in\mathbb{R}$. An equivalent condition is $f(\lambda x+(1-\lambda)y)\le f(x)\vee f(y), 0\le\lambda\le 1$. That is why such functions are also called level convex.

Now let $\zeta(\cdot)$ be arbitrary. We have, using this inequality with $\dot{\xi} = \zeta$,

$$\begin{aligned} u(t,x) = \inf_{\zeta} g(\xi(t)) \vee \operatorname*{ess\,sup}_{\tau\in[0,t]} H^{\#}(\zeta(\tau)) &\geq \inf_{\zeta} g(\xi(t)) \vee H^{\#}\Big(\frac{1}{t}\int_0^t \dot{\xi}(s)\, ds\Big) \\ &= \inf_{\zeta} g(\xi(t)) \vee H^{\#}\Big(\frac{\xi(t)-x}{t}\Big) \\ &\geq \inf_{z} g(z) \vee H^{\#}\Big(\frac{z-x}{t}\Big). \end{aligned}$$

That's how we get the Lax formula for (1.5). This was the entrance into consideration of quasiconvex functions and exact formulas for various nonlinear equations.

I record here the various definitions of quasiconvex conjugates for functions.

1.3. Quasiconvex Conjugates

Given a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ define

1. $f^{\%}(r,p) = \sup\{p \cdot x \wedge r - f(x) \mid x \in \mathbb{R}^n\}$, $f^{\%\%}(x) = \sup\{(p\cdot x \wedge r - f^{\%}(r,p)) \mid r \in \mathbb{R}, p \in \mathbb{R}^n\}$.
2. $f^{\sharp}(r,p) = \sup\{p\cdot x - f(x) \mid x \in \{f(x) \leq r\}\}$, $f^{\sharp\sharp}(x) = \sup\{(p\cdot x - f^{\sharp}(r,p)) \wedge r \mid r \in \mathbb{R}, p \in \mathbb{R}^n\}$.
3. $f^{\#}(r,p) = \sup\{p \cdot x \mid x \in \{f(x) \leq r\}\}$, $f^{\#\#}(x) = \inf\{r \mid \sup_{p\in\mathbb{R}^n} p \cdot x - f^{\#}(r,p) \leq 0\}$.

A lower semicontinuous function f is quasiconvex if and only if $f = f^{\%\%}$, $f = f^{\sharp\sharp}$, or $f = f^{\#\#}$.

1.4. Hopf from Lax

In general if we switch the convexity from the Hamiltonian to the initial data we seek a Hopf formula for the solution. This involves invoking a minimax theorem. We will formally illustrate the derivation of the Hopf formula for (1.5) for quasiconvex functions g from the Lax formula. Begin with the Lax formula:

$$\begin{aligned} u(t,x) &= \inf_{z} g(z) \vee H^{\#}\Big(\frac{z-x}{t}\Big) \\ &= \inf_{z} g(z) \vee \inf\{\gamma \mid \sup_{p}(p\cdot z - H(\gamma,p)) \leq 0\} \\ &= \inf_{z} \inf\{\gamma \mid g(z) \leq \gamma \text{ and } \sup_{p}(p\cdot z - H(\gamma,p)) \leq 0\} \\ &= \inf\{\gamma \mid \sup_{p} p\cdot x - g^{\#}(\gamma,p) - tH(\gamma,p) \leq 0\} \\ &= (g^{\#}(\gamma,p) + tH(\gamma,p))^{\#}. \end{aligned}$$

We have formally switched infimum and supremum in the fourth line. However unjustified, this does lead to the Hopf formula for the equation (1.5). It should be mentioned that no published proof justifying this derivation is known to me.

Theorem 2. *If g is quasiconvex, $\gamma \mapsto H(\gamma,p)$ is nondecreasing, and $p \mapsto H(\gamma,p)$ is positively homogeneous degree one, then the unique viscosity solution of* (1.5) *is given by the Hopf formula,*

$$u(t,x) = (g^{\#}(\gamma,p) + tH(\gamma,p))^{\#}.$$

This formula still holds if g is merely quasiconex and lower semicontinuous and bounded from below by a function with linear growth.

Refer to [1] for more details and a proof of this and more general results.

1.5. Hopf–Lax Formulas Obstacle Problems

In 1995, A. Subbotin in his classic book [13] considered the upper obstacle problem

$$\min\{u_t + H(Du), g(x) - u\} = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n, \quad u(T,x) = g(x), \tag{1.6}$$

with g a convex function. We switch back to terminal value problems to match Subbotin's formula. Subbotin proved, by using his theory of minimax solutions, that the function u is given by a Hopf formula

$$u(t,x) = (g^* - (T-t)(H \wedge 0))^* = \sup_{y \in \mathbb{R}^n} y \cdot x - g^*(y) + (T-t)(H(y) \wedge 0),$$

or, equivalently,

$$u(t,x) = \sup_{y \in \mathbb{R}^n} \big(y \cdot x - g^*(y) + (T-t)H(y)\big) \wedge g(x).$$

Several questions arise concerning this obstacle problem. The first is whether there is a Lax formula for (1.6). The answer was affirmatively given in [4].

Theorem 3. *Assume $H\colon \mathbb{R}^n \to \mathbb{R}$ is convex and coercive and $g\colon \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous. The unique uniformly continuous viscosity solution of* (1.6) *is given by*

$$u(t,x) = \sup_{z \in \mathbb{R}^n} \inf_{t \le \tau \le T} \Big(g(z) - (\tau - t)H^*\Big(\frac{z-x}{\tau - t}\Big)\Big) = \sup_{z \in \mathbb{R}^n} \inf_{t \le \tau \le T} \big(g(x + z(\tau - t)) - (\tau - t)H^*(z)\big).$$

The key to the proof of this theorem is the representation of the unique solution of (1.6) as

$$u(t,x) = \sup_{\zeta \in \mathcal{Z}[t,T]} \inf_{t \le \tau \le T} g(\xi(\tau)) - \int_t^\tau H^*(\zeta(s))\, ds,$$

where $\dot{\xi} = \zeta,\ \xi(t) = x$. See [4] and [5] for details and the proof.

1.6. Hopf–Lax with Quasiconvex Obstacle

Next we consider the obstacle problem but with quasiconvex obstacle in the Hopf formula. As already noted above, the class of quasiconvex functions is vastly larger than convex functions and quasiconvex functions need not even be continuous, so this is a nontrivial extension of the class of possible obstacles. On the other hand such an extension will require more assumptions on H, which we allow to depend on u and Du.

Let $H\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be continuous with $\gamma \mapsto H(\gamma, p)$ nondecreasing and $p \mapsto H(\gamma, p)$ homogeneous, degree one. If $g\colon \mathbb{R}^n \to \mathbb{R}$ is continuous and quasiconvex, the unique viscosity solution of the obstacle problem

$$\min\{u_t + H(u, Du), g - u\} = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n, \quad u(T,x) = g(x), \quad x \in \mathbb{R}^n,$$

is given by the Hopf formula

$$\begin{aligned} u(t,x) &= \big(g^{\%}(\gamma,p) - (T-t)H(\gamma,p) \wedge 0\big)^{\%}(x) \\ &= \sup_{\gamma \in \mathbb{R}, p \in \mathbb{R}^n} \big(p \cdot x - g^{\%}(\gamma,p) + H(\gamma,p) \wedge 0(T-t)\big) \wedge \gamma. \end{aligned}$$

Here the quasiconvex conjugates used are

$$g^{\%}(\gamma,p) = \sup_{x \in E(\gamma,g)} p \cdot x - g(x), \quad g^{\%\%}(x) = \sup_{\gamma \in \mathbb{R}, p \in \mathbb{R}^n} \big(p \cdot x - g^{\%}(\gamma,p)\big) \wedge \gamma.$$

However, it can be shown that the use of any of the quasiconvex conjugates can be used. The proof of this formula can be found in [9].

1.7. An Extension to Time Dependent Obstacle

Returning to the convex case, we extend the formula derived by Subbotin in [13] to the case when the **lower** obstacle is $h : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ with $x \mapsto h(t,x)$ uniformly Lipschitz continuous and convex for each $t \in [0,T]$. We assume $g(x) \geq h(t,x)$. Then, the solution of $u(T,x) = g(x)$,

$$\max\big\{u_t + H(Du),\, h(t,x) - u\big\} = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n, \quad u(T,x) = g(x),$$

is

$$u(t,x) = \sup_{y \in \mathbb{R}^n} \big(y \cdot x - g^*(y) + H(y)(T-t)\big) \vee \big(\sup_{t \leq \tau \leq T} y \cdot x - h^*(\tau, y) + H(y)(\tau - t)\big).$$

Refer to [5] for a proof of this assertion. Notice that it reduces to Subbotin's formula when $h(t,x) = g(x)$.

1.8. Connection with Differential Games with Stopping Times

Many of the results mentioned above may be proved using the representation of u as the value function for a particular differential game constructed using the given Hamiltonian. Here is how the approach works for the time dependent obstacle.

Let L_g and L_H denote, respectively the Lipschitz constant for g and H. Using a representation due to Bardi–Evans [3], by writing

$$H(p) = \max_{|y| \leq L_g} \min_{|z| \leq L_H} p \cdot z + H(y) - y \cdot z$$

the obstacle equation $\max\{u_t + H(Du), h - u\} = 0$ becomes

$$\max\big\{u_t + \max_{|y| \leq L_g} \min_{|z| \leq L_H} D_x u \cdot z + H(y) - y \cdot z,\, h(t,x) - u\big\} = 0,$$

and $u(T,x) = g(x)$. The function u is the (lower) value function of a differential game with stopping times given by

$$u(t,x) = \sup_{t \leq \tau \leq T} \inf_{\Delta \in \mathcal{Z}[t,T]} \sup_{\eta \in Y[t,T]} g(\xi(\tau))\chi_{\tau=T} + h(\tau, \xi(\tau))\chi_{\tau<T} + \int_t^{\tau \wedge T} H(\eta(s)) - \eta(s) \cdot \zeta(s)\, ds,$$

where $\zeta(s) = \Delta[\eta](s)$ and $\dot{\xi}(s) = \zeta(s), t < s \leq T, \xi(t) = x \in \mathbb{R}^n$. Using Jensen's inequality we can evaluate this to

$$u(t,x) = \sup_{y \in \mathbb{R}^n} \big(y \cdot x - g^*(y) + H(y)(T-t)\big) \vee \big(\sup_{t \leq \tau \leq T} y \cdot x - h^*(\tau, y) + H(y)(\tau - t)\big).$$

If instead of a lower obstacle, one had an upper obstacle, we simply replace $\sup_{t \leq \tau \leq T}$ with $\inf_{t \leq \tau \leq T}$. The theory of differential games with stopping times was introduced by Besoussan and Friedman in [11]. Refer to [9] for the details and proofs of the assertions in this section.

1.9. Double Obstacle Problem

It is also possible to consider problems in which there is a lower and upper obstacle, i.e., a double obstacle problem as in [9]. The importance of doing so is that double obstacle problems arise in many applications but in particular the new field of reach-avoid control in the next section. The following theorem gives the Hopf formula when both obstacles are convex.

Theorem 4. *Assume* $H : \mathbb{R}^n \to \mathbb{R}$ *is continuous and* $g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2,$ *are uniformly Lipschitz continuous and convex with* $g_1 \le g_2$. *The unique uniformly continuous viscosity solution of*

$$\max\big\{\min\{u_t + H(Du),\ g_2 - u\}, g_1 - u\big\} = 0,\ (t,x) \in [0,T) \times \mathbb{R}^n,\ u(T,x) = g_2(x),\ x \in \mathbb{R}^n, \tag{1.7}$$

is given by

$$u(t,x) = g_1(x) \vee \sup_{y \in \mathbb{R}^n} \big(y \cdot x - g_2^*(y) + H(y) \wedge 0(T-t)\big), \quad (t,x) \in [0,T] \times \mathbb{R}^n. \tag{1.8}$$

Again the proof of this theorem is carried out with the representation of the solution of (1.7) as the lower value of a differential game with stopping times. Since $u(T,x) = g_2(x)$ the solution of (1.7) is given by (see [11, Theorem 2.4] for the proof),

$$u(t,x) = \inf_{t \le \tau \le T} \sup_{t \le \sigma \le T} \inf_{\Delta} \sup_{\eta} g_1(\xi(\sigma))\chi_{\sigma<\tau} + g_2(\xi(\tau))\chi_{\tau\le\sigma} + \int_t^{\tau\wedge\sigma} H(\eta(s)) - \eta(s)\cdot\zeta(s)\, ds$$

with $\dot{\xi}(s) = \zeta(s),\ t < s \le T,\ \xi(t) = x \in \mathbb{R}^n$. It is now a task to evaluate this so that it is no longer over controls, but finite dimensional optimization. That leads to (1.8). It is an open problem to develop a direct proof that (1.8) is the viscosity solution of (1.7).

Many if not all the Hopf and Lax formulas may be extended to the case when $H = H(t,u,Du)$, i.e., when we have time dependence. In the basic cases this has already been done (see [14] and the references there). For many of the problems mentioned here, such as the double obstacle problem, allowing time dependence in the Hamiltonian or in the obstacles as done in the single obstacle case is open. On the other hand, it is very doubtful that explicit formulas in finite dimensions can be found for Hamiltonians with spatial dependence in any sort of generality.

2. The Reach-Avoid Control Problem

A reach-avoid control problem has a state trajectory given by $\dot{\xi} = f(\tau, \xi(\tau), \zeta(\tau)), t < \tau, \xi(t) = x \in \mathbb{R}^n$, in which the control is $\zeta(\cdot)$ with the goal of attaining a target $\mathcal{T} = \{x \in \mathbb{R}^n \mid g_u(x) \le 0\}$ while avoiding a given set $\mathcal{A} = \{x \in \mathbb{R}^n \mid g_\ell(x) > 0\}$ for all times $t \le \tau \le T$.

The payoff of such a control problem is given by

$$P(\zeta) = \inf_{t \le \tau \le T} g_u(\xi(\tau)) \vee \sup_{t \le \sigma \le \tau} g_\ell(\xi(\sigma)),$$

and the value function is $v(t,x) = \inf_\zeta P(\zeta)$.

The reach-avoid set is

$$RA = \{(t,x) \in [0,T] \times \mathbb{R}^n \mid v(t,x) \le 0\}.$$

The assumption $g_\ell \le g_u$ is natural since then $\mathcal{T} = \{x \mid g_u(x) \le 0\} \subset \mathcal{A}^c = \{x \mid g_\ell(x) \le 0\}$.

Variations on the problem include attaining the target at time T, rather than at any time up to T, in which case the payoff is $P(\zeta) = g_u(\xi(T)) \vee \sup_{t \le \sigma \le T} g_\ell(\xi(\sigma))$.

2.1. The Hamilton–Jacobi Equation for Reach-Avoid Problems

For the control problem, the following characterizes v with terminal condition $v(T,x) = g_u(x)$, (see Fisac–Chen–Tomlin–Sastry [12])

$$\max\big\{\min\{v_t + \min_z D_x v \cdot f(t,x,z), g_u - v\}, g_\ell - v\big\} = 0.$$

If g_u, g_ℓ depend on (t, x, z) and more generally, if $f = f(t, x, y, z), g_u = g_u(t, x, y, z)$ and $g_\ell = g_\ell(t, x, y, z)$, we may consider a differential reach-avoid game which was introduced and studied in [6]. This paper [6] considers the differential game reach-avoid game in which the targets and obstacles can move. This allows models of systems in which the target and avoidance set may act antagonistically.

3. Appendix: Characterization of Quasiconvex Functions

A function $u : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex if $E_\alpha = \{x \in \mathbb{R}^n \mid u(x) \le \alpha\}$ is a convex set for each $\alpha \in \mathbb{R}$. An equivalent characterization is

$$u(\lambda x + (1-\lambda)y) \le \max\{u(x), u(y)\} \;\; \forall\, 0 \le \lambda \le 1, \quad x, y \in \mathbb{R}^n.$$

If u is differentiable, a first order condition is

$$u(x) \le u(y) \implies Du(y) \cdot (x - y) \le 0.$$

The distinction in first order conditions between convex and quasiconvex functions is evident. The question is what is a second order condition for quasiconvexity?

3.1. Second Order Condition for Quasiconvexity

Let $\Omega \subset \mathbb{R}^n$ be a convex set. For a continuous function $u : \Omega \to \mathbb{R}$

- $u : \Omega \to \mathbb{R}$ is convex iff $D^2 u \succeq 0$, i.e., $\min_{|y|=1} yD^2uy^T \ge 0$ in the viscosity sense;
- if u is quasiconvex, in the viscosity sense, $L(u) = \min\{yD^2uy^T \mid |y| = 1, y \cdot Du = 0\} \ge 0$. The converse is false in general, but there is a partial converse: If u is upper semicontinuous and $L(u) > 0$, then u is quasiconvex.

The equation $L(u) = h$ arises in a tug-of-war game due to Kohn–Serfaty involving the exit time from a domain Ω. The dynamic programming principle for this game leads to

$$u(x) = \min_{|y|=1} \max_{\{b=\pm 1\}} u(x + \sqrt{2}\varepsilon b y) - \varepsilon^2.$$

Expand this to second order to get

$$u(x) = \min_{|y|=1} \max_{\{b=\pm 1\}} u(x) + \sqrt{2}\varepsilon b y D_x u + \varepsilon^2 y D^2 u y^T - \varepsilon^2 + o(\varepsilon^2)$$

or

$$0 = \min_{|y|=1} \sqrt{2}\varepsilon |yD_x u| + \varepsilon^2 y D^2 u y^T - \varepsilon^2 + o(\varepsilon^2).$$

Divide by ε^2 and send $\varepsilon \to 0$ to get $\displaystyle\min_{\{|y|=1, yDu=0\}} yD^2uy^T - 1 = 0.$

Equations of the form $L(u) = h(x)$ arise naturally in several contexts. One such context is stochastic control in L^∞. A basic example is the following.

The dynamics are $d\xi = \sqrt{2}\eta(t)dW(t),\ 0 < t \le \tau, \xi(0) = x,\ \tau = \tau_x$ is exit time from Ω. The controls are $\eta, |\eta| = 1$. Given a running cost function $h(x) \ge C > 0$, we define the value function by

$$u(x) = \inf_{\eta} \operatorname{ess\,sup}_\omega g(\xi(\tau_x)) - \int_0^{\tau_x} h(\xi(s))\, ds.$$

Then, it is shown in [8] that the function u is the unique continuous viscosity solution of $L(u) = h, x \in \Omega, u = g, x \in \partial O$.

The following proves viscosity solutions are unique.

Theorem 5 [7; 8]. *If u is an upper semicontinuous subsolution of $L(u) - h(x) \ge 0$ and v is a lower semicontinuous supersolution of $L(v) - h(x) \le 0$, where $h(x) \ge C > 0$, then $u \le v$.*

3.2. Quasiconvex Envelope of a Given Function

The following theorem gives a way to construct the greatest quasiconvex minorant of a given function $g : \Omega \to \mathbb{R}$.

Theorem 6 [7]. *The unique quasiconvex viscosity solution of* $\min\{g-u, L(u)\} = 0$ *in* Ω *with* $u = g$ *on* $\partial\Omega$*, is given by* $u = g^{\#\#}$*, the greatest quasiconvex minorant of* g.

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